LECTURE 20 FRIDAY OCT 18

We continue to discuss how the algebra $\mathbb{A}_q$ is related to the quantum group $U_q = U_q(sl_2)$.

In Prop 52 we gave an $\mathbb{F}$-algebra hom $\mathbb{A}_q \to U_q$. We now consider the image of $\mathbb{A}_q$ in U_q.

Define a set of 3-tuples

$$\mathcal{S} = \{ (i,j,k) \mid i,k \in \mathbb{N},\ j \in \mathbb{Z} \}$$

$\mathbb{Z}$ = integers $\qquad \mathbb{N} = \{0,1,2,\ldots\}$

It is known (ex) that the following is a basis for the $\mathbb{F}$-vector space U_q:

$$x^i y^j z^k \qquad (i,j,k) \in \mathcal{S}$$

Using this, one finds that the map

$\Delta_q \to U_q$ in Prop 52 is injective. Via this

injection we identify Δ_q with its image.

Note that

$$\frac{xy-yx}{q-q^{-1}} = q(1-xy) = q^{-1}(1-yx)$$

Call this n_z. Similarly define n_x, n_y.

Thus

$$A = i(x-y) + n_z$$

$$B = i(y-z) + n_x \qquad i^2 = -1$$

$$C = i(z-x) + n_y$$

LEM 53 $\forall\, r, s, t \in \mathbb{N}$

$\exists$ $0 \neq \alpha_{rst} \in \mathbb{F}$ such that

$$A^r B^s C^t - \alpha_{rst}\, x^{r+t} y^{r+s} z^{s+t}$$

$$\in \text{Span}\{ x^i y^j z^k \mid i, j, k \in \mathbb{N}, \quad i+j+k < 2(r+s+t) \} \quad (*)$$

pf

$$A^r B^s C^t = \big(i(x-y) + q(1-xy) \big)^r \big(i(y-z) + q(1-yz) \big)^s$$
$$\big(i(z-x) + q(1-zx) \big)^t$$

Reduce RHS using the relations in the equitable presentation of U_q.

Result is a nonzero scalar mult of $x^{r+t} y^{r+s} z^{s+t}$ plus terms in $*$.

$\square$

Define a set of 3-tuples

$$S = \{ (i,j,k) \mid i,j,k \in \mathbb{N},\ i+j+k \text{ even},\ i \le j+k,\ j \le k+i,\ k \le i+j \}$$

$$= \{ (r+t,\ r+s,\ s+t) \mid r,s,t \in \mathbb{N} \}.$$

View $S \subseteq \mathcal{S}$

Prop 54 The following is a basis for a complement of $\mathbb{A}_q$ in U_q:

$$x^i y^j z^k \qquad (i,j,k) \in \mathcal{S} \setminus S \qquad ☆$$

pf ☆ is a basis for a subspace of U_q, denoted W. Show

$$U_q = \mathbb{A}_q + W \qquad \text{(dir sum)}$$

Consider our basis for U_q:

$$x^i y^j z^k \qquad (i,j,k) \in \mathcal{S}$$

Write this basis as

$$\{x^i y^j z^k \mid (i,j,k) \in S\} \cup \{x^i y^j z^k \mid (i,j,k) \in \mathcal{S} \setminus S\}$$

$$= \{x^{r+t} y^{r+s} z^{s+t} \mid r,s,t \in \mathbb{N}\} \cup \{x^i y^j z^k \mid (i,j,k) \in \mathcal{S} \setminus S\} \qquad *$$

Recall $\mathbb{A}_q$ has basis

$$A^r B^s C^t \qquad r,s,t \in \mathbb{N}$$

Write the vectors

$$\{A^r B^s C^t \mid r,s,t \in \mathbb{N}\} \cup \{x^i y^j z^k \mid (i,j,k) \in \mathcal{S} \setminus S\} \qquad **$$

in the basis (*). By LEM 53 the coef matrix is upper triangular with all diag entries non 0. Therefore (**) is a basis for U_q. It follows that

$$U_q = \mathbb{A}_q + W \qquad (ds)$$

□

Recall the central element Ω of Δ_q.

Next goal: show Ω is central in U_q

Define $\Lambda \in U_q$ by

$$\Lambda = (q-q^{-1})^2 ef + q^{-1}k + qk^{-1}$$

One checks

Λ is central in U_q

Call Λ the <u>Casimir element</u> of U_q

In the equit presentation Λ looks as follows.

$$\begin{aligned}\Lambda &= qx + q^{-1}y + qz - qxyz \\ &= qy + q^{-1}z + qx - qyzx \\ &= qz + q^{-1}x + qy - qzxy\end{aligned}$$

and

$$\begin{aligned}\Lambda &= q^{-1}x + qy + q^{-1}z - q^{-1}zyx \\ &= q^{-1}y + qz + q^{-1}x - q^{-1}xzy \\ &= q^{-1}z + qx + q^{-1}y - q^{-1}yxz\end{aligned}$$

the elements Λ, n_x, n_y, n_z satisfy the following identities (all readily checked):

$$xy = 1 - q^{-1} n_z \qquad (+\,CP)$$

$$yx = 1 - q\, n_z$$

$$\frac{xy - yx}{q - q^{-1}} = n_z, \qquad (+\,CP)$$

$$\frac{qyx - q^{-1}xy}{q - q^{-1}} = 1 - (q + q^{-1})\, n_z$$

$$x\, n_y = q^{2} n_y\, x \qquad (+\,CP)$$

$$x\, n_z = q^{-2} n_z\, x$$

$$n_x\, x = \Lambda - q y - q^{-1} z \qquad (+\,CP)$$

$$x\, n_x = \Lambda - q^{-1} y - q z$$

$$\frac{x n_x - n_x x}{q - q^{-1}} = y - z \qquad (+ CP)$$

$$\Lambda = \frac{q x n_x - q^{-1} n_x x}{q - q^{-1}} + (q + q^{-1}) z \qquad (+ CP)$$

$$\Lambda = \frac{q n_x x - q^{-1} x n_x}{q - q^{-1}} + (q + q^{-1}) y$$

$$n_x n_y = 1 - q^{-1} \Lambda z + q^{-2} z^2 \qquad (+ CP)$$

$$n_y n_x = 1 - q \Lambda z + q^2 z^2$$

$$\frac{q n_x n_y - q^{-1} n_y n_x}{q - q^{-1}} = 1 - z^2 \qquad (+ CP)$$

— o —

Prop 55

$$\Omega = (q + q^{-1})^2 - \Lambda^2$$

Moreover Ω is central in U_q.

pf Recall

$$\Omega = q ABC + q^2 A^2 + q^{-2} B^2 + q^2 C^2$$

In RHS elim A, B, C using

$A = i(x-y) + n_z$, $B = i(y-z) + n_x$, $C = i(z-x) + n_y$

and simplify using the above identities. □

Some open problems concerning $\mathbb{A}_q$

Recall the action of $PSL_2(\mathbb{Z})$ on $\mathbb{A}_q$:

ρ sends $A \to B \to C \to A$

σ sends $A \leftrightarrow B$

It seems this $PSL_2(\mathbb{Z})$ action does not extend to U_q, since the extended ρ would send

$$x \to y \to z \to x$$
$$y^{-1} \to ?$$

Problem 56 Find a "completion" of U_q that has a natural $PSL_2(\mathbb{Z})$ action.

Note that $\sigma\rho$ fixes A

Conjecture 57 The group $\{ g \in PSL_2(\mathbb{Z}) \mid g(A) = A \}$ is generated by $\sigma\rho$.

Let $\mathcal{O} \subseteq \mathbb{A}_q$ denote the orbit of $PSL_2(\mathbb{Z})$ that contains A.

For the moment, view $\mathbb{A}_q$ as a Lie algebra over $\mathbb{F}$ with Lie bracket $[r,s] = rs - sr$

Let $\mathcal{L}$ denote the Lie subalgebra of $\mathbb{A}_q$ generated by A, B, C.

Conjecture 58 $\mathcal{O}$ is a basis for $\mathcal{L}$.

— o —

Observe $\exists$ $\mathbb{F}$-algebra hom $\mathbb{A}_q \to \mathbb{F}$ that sends

$$A \to 0, \quad B \to 0, \quad C \to 0$$

Let $\tilde{\mathbb{A}}_q$ denote the kernel. One checks

$$\mathbb{A}_q = \tilde{\mathbb{A}}_q + \mathbb{F}1 \qquad (ds)$$

Conjecture 59 The following is a basis for the $\mathbb{F}$-vector space $\widetilde{\mathbb{A}}_q$:

$$\Omega^i, \quad \{u^i \mid u \in \Theta\} \qquad i = 1, 2, 3, \ldots$$

— o —

Problem 60 View $\mathbb{A}_q$ as a module for $PSL_2(\mathbb{Z})$. Describe the irred. $PSL_2(\mathbb{Z})$ submodules. Is $\mathbb{A}_q$ a dir sum of irred $PSL_2(\mathbb{Z})$ submodules?

Problem 61 Find all the 2-sided ideals of $\mathbb{A}_q$. Which of these are $PSL_2(\mathbb{Z})$-invariant?

Problem 62 Find all the $PSL_2(\mathbb{Z})$-inv subalgebras of $\mathbb{A}_q$.

Next topic: the split decomposition of a TD pair.

Until further notice

$\mathbb{F}$ = any field

V = vector space over $\mathbb{F}$ with finite pos dim

For an integer $d \geq 0$

A <u>decomposition of V of length d</u> is a sequence $\{U_i\}_{i=0}^d$ of subspaces of V s.t. $U_i \neq 0$ for $0 \leq i \leq d$

and $V = \sum_{i=0}^{d} U_i$ (dir sum)

We set $U_{-1} = 0$, $U_{d+1} = 0$

We fix a TD system on V:

$$\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta\right)$$

with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^\delta$

DEF 63 $\forall$ integers i, j define

$$V_{ij} = \left(E_0^*V + E_1^*V + \cdots + E_i^*V\right) \cap \left(E_jV + E_{j+1}V + \cdots + E_dV\right)$$

Interp sum on left to be 0 if $i < 0$ and V if $i > \delta$

Interp sum on right to be 0 if $j > d$ and V if $j < 0$

LEM 64 We have

(i) $V_{i0} = E_0^*V + \cdots + E_i^*V$ $\quad 0 \le i \le \delta$

(ii) $V_{\delta j} = E_jV + \cdots + E_dV$ $\quad 0 \le j \le d$

pf clear $\square$

LEM 65 For $0 \le i \le \delta$ and $0 \le j \le d$,

(i) $(A - \theta_j I) V_{ij} \subseteq V_{i+1, j+1}$

(ii) $A V_{ij} \subseteq V_{ij} + V_{i+1, j+1}$

(iii) $(A^* - \theta_i^* I) V_{ij} \subseteq V_{i-1, j-1}$

(iv) $A^* V_{ij} \subseteq V_{ij} + V_{i-1, j-1}$

pf (i) We have

$$(A - \theta_j I)(E_0^* V + \cdots + E_i^* V) \subseteq E_0^* V + \cdots + E_{i+1}^* V$$

and

$$(A - \theta_j I)(E_j V + \cdots + E_d V) = E_{j+1} V + \cdots + E_d V$$

(ii) By (i)

(iii), (iv) Sim. $\square$