Fall 2013

Math 846

TRIDIAGONAL PAIRS and Related Topics

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MWF 12:05 – 12:55.

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Lecture 1 Wednesday Sept 4 9/4/13

Introduction

$\mathbb{F}$: a field

V: a vector space over $\mathbb{F}$ with finite positive dim

$$\mathrm{End}(V) = \{A : V \to V \mid A \text{ is } \mathbb{F}\text{-linear}\}$$

For $A \in \mathrm{End}(V)$ and a subspace $W \subseteq V$

W is an eigenspace for A whenever $W \neq 0$ and $\exists\ \theta \in \mathbb{F}$ such that

$$W = \{v \in V \mid Av = \theta v\}$$

A is diagonalizable whenever V is spanned by the eigenspaces of A

Def A <u>tridiagonal pair</u> (a <u>TD pair</u>) on V is an ordered pair A, A^* of elements in $End(V)$ such that

(i) Each of A, A^* is diagonalizable

(ii) $\exists$ an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$, $V_{d+1} = 0$

(iii) $\exists$ an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$

(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$

We say A, A^* is <u>over</u> $\mathbb{F}$. Call V the <u>underlying vector space</u>

Note A^* does not mean conjugate-transpose

V^* does not mean dual space

A, A^* are arbitrary linear trans that satisfy (i)-(iv)

Some basic facts about TD pairs

(proved later)

Given TD pair A, A^* on V as in Def 1

It turns out

$$d = \delta$$

Call this the <u>diameter</u> of the pair

Fact I

For $0 \le i \le d$ let

θ_i = eigenvalue of A for V_i

θ_i^* = eigenvalue of A^* for V_i^*

Then

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \le i \le d-1$

<u>Note</u> Let $\overline{\mathbb{F}}$ = algebraic closure of $\mathbb{F}$

The "most general" solution to above recurrence is

$$\theta_i = a + bq^i + cq^{-i} \qquad 0 \le i \le d$$

$$\theta_i^* = a^* + b^* q^i + c^* q^{-i}$$

where $q, a, b, c, a^*, b^*, c^* \in \overline{\mathbb{F}}$, $q \ne 0, \pm 1$

Fact II

there exist scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in $\mathbb{F}$ such that

$$[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] = 0,$$

$$[A^*,\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta A] = 0$$

where

$$[x, y] = xy - yx$$

"TD relations"

<u>Note</u> Assume $\gamma = \gamma^* = 0$, $\delta = \delta^* = 0$, $\beta = q^2 + q^{-2}$, $0 \neq q \in F$, $q \neq \pm 1$

TD rels become

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0,$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0$$

where $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

"cubic q-Serre relations"

Fact III

For $0 \le i \le d$ define

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$$

then

$$V = \sum_{i=0}^{d} U_i \qquad \text{(dir sum)}$$

Also

$$(A - \theta_i I) U_i \subseteq U_{i+1} \qquad 0 \le i \le d-1$$

$$(A - \theta_d I) U_d = 0$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \qquad 1 \le i \le d$$

$$(A^* - \theta_0^* I) U_0 = 0$$

"Split decomposition"

Fact IV

For $0 \le i \le d$

$$\dim V_i = \dim V_i^* \qquad (= \rho_i)$$

Moreover the sequence $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal, ie

$$\rho_i = \rho_{d-i} \qquad 0 \le i \le d$$

$$\rho_{i-1} \le \rho_i \qquad 1 \le i \le d$$

Also

$$\rho_i \le \rho_0 \binom{d}{i} \qquad 0 \le i \le d$$

binomial coef

the sequence $\{\rho_i\}_{i=0}^{d}$ is called the <u>shape</u>

A, A^* is called <u>sharp</u> whenever $\rho_0 = 1$

FACT V

If $\mathbb{F}$ is algebraically closed then

A, A^* is sharp

FACT VI

Assume A, A^* is sharp.

Then there exists a nondegenerate, symmetric bilinear form $\langle , \rangle$ on V

such that both

$$\langle Au, v \rangle = \langle u, Av \rangle$$

$$\langle A^*u, v \rangle = \langle u, A^*v \rangle$$

for all $u, v \in V$.

A, A^* is called a Leonard pair whenever $\rho_i^* = 1$ $(0 \le i \le d)$

Fact VII

The Leonard pairs are classified up to isomorphism.

They are in bijection with a family of orthogonal polynomials consisting of:

q-Racah
q-Hahn
dual q-Hahn

q-Krawtchouk
dual q-Krawtchouk
affine q-Krawtchouk
quantum q-Krawtchouk

Racah
Hahn
dual Hahn

Krawtchouk

Bannai/Ito

Orphans (char $\mathbb{F} = 2$ only)

This family is the terminating branch of the Askey scheme of orthogonal polynomials

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FACT VIII

the sharp TP pairs are classified up to 150.

Connections

Tridiagonal Pairs

Linear algebra:
- split decomposition
- Askey-Wilson relations
- Tridiagonal relations

Combinatorics:
- Q-polynomial distance-regular graphs
- uniform posets

Lie algebra:
- sl_2,
- sl_2-loop algebra
- Onsager algebra $\mathcal{O}$
- tetrahedron algebra $\boxtimes$

quantum algebra:
$U_q sl_2$, $U_q \widehat{sl_2}$, $\boxtimes_q$, $\mathcal{O}_q$

Special functions + Orthogonal polynomials:
- q-Racah polys + relatives in Askey scheme
- (basic) hypergeometric series
- Leonard pairs

Double affine Hecke algebra of type $(C_1^\vee, C_1)$

Comment on Linear algebra

Let V denote a vector space over $\mathbb{F}$ with finite pos dimension.

Assume $A \in \mathrm{End}(V)$ is diagonalizable

Let $\{V_i\}_{i=0}^{d}$ denote an ordering of the eigenspaces of A. So $V = \sum_{i=0}^{d} V_i$ (dir sum)

For $0 \le i \le d$ let θ_i denote the eigenvalue of A for V_i

For $0 \le i \le d$ define $E_i \in \mathrm{End}(V)$ such that

$$(E_i - I) V_i = 0$$

$$E_i V_j = 0 \quad \forall j \ne i \quad (0 \le j \le d)$$

So E_i is the projection onto V_i

Observe

$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le d)$$

$$I = \sum_{i=0}^{d} E_i$$

$$A = \sum_{i=0}^{d} \theta_i E_i$$

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \le i \le d)$$

$$V_i = E_i V \qquad 0 \le i \le d$$

Let M denote the subalgebra of $\mathrm{End}(V)$ generated by A. Then each of

$$\{A^i\}_{i=0}^{d} \qquad \{E_i\}_{i=0}^{d}$$

is a basis for the $\mathbb{F}$-vector space M.

Moreover

$$\prod_{i=0}^{d} (A - \theta_i I) = 0.$$

We call $\{E_i\}_{i=0}^{d}$ the <u>primitive idempotents</u> of A