LECTURE 19 Wednesday Oct 16

pf of thm 46

Assume θ_i, θ_i^* sat (i)–(iii).

Show $\exists$ totally bipartite TD system Φ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$

Define

$$c_i = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \qquad 1 \le i \le d-1$$

$$c_d = \theta_0$$

$$b_i = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i-1}^*}{\theta_{i+1}^* - \theta_{i-1}^*} \qquad 1 \le i \le d-1$$

$$b_0 = \theta_0$$

One checks

$$c_i \neq 0 \qquad 1 \le i \le d,$$

$$b_i \neq 0 \qquad 0 \le i \le d-1$$

Define

$$A = \begin{pmatrix} 0 & b_0 & & & \mathbf{0} \\ c_1 & 0 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ \mathbf{0} & & & c_d & 0 \end{pmatrix}$$

$$A^* = \operatorname{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$$

$$E_i^* = \operatorname{diag}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \qquad 0 \le i \le d$$

So A is irred tridiag

Define v.s. $V = \mathbb{F}^{d+1}$ (col vectors)

Identify

$$\mathrm{End}(V) = \mathrm{Mat}_{d+1}(\mathbb{F})$$

For $0 \le i \le d$ define

$$V_i^* = E_i^* V$$

So

V_i^* is eigenspace for A^* with eigval θ_i^*

and

$$A V_i^* \subseteq V_{i-1}^* + V_{i+1}^* \qquad 0 \le i \le d$$

where $V_{-1}^* = 0$, $V_{d+1}^* = 0$

Since A is irred tridiag

each eigenspace of A has dim 1

(Caution: possibly A is not diagonalizable)

By LEM 6

V is irred as a module for A, A^*

By matrix mult

$$A^2A^* - \beta AA^*A + A^*A^2 = \rho A^*$$

where ρ is from L21

For $0 \le i \le d$ define

$$V_i = \{ v \in V \mid Av = \theta_i v \}$$

so $\dim V_i \le 1$

We will show

$$\dim V_i = 1 \qquad 0 \le i \le d$$

and

$$A^* V_i \subseteq V_{i-1} + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

Define
$$v_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in V$$

claim 1 $\quad v_0$ is a basis for V_0

pf cl $\quad$ Obs

$$c_i + b_i = \theta_0 \qquad 0 \le i \le d$$

where $\quad c_0 = 0, \quad b_d = 0$

therefore $\quad Av_0 = \theta_0 v_0$ ✓

Define
$$v_1 = \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \theta_d^* \end{pmatrix} \in V$$

claim 2 $\quad v_1$ is a basis for V_1. Moreover
$$A^* v_0 = v_1$$

pf $\quad$ One checks

$$c_i \theta_{i-1}^* + b_i \theta_{i+1}^* = \theta_1 \theta_i^* \qquad 1 \le i \le d-1,$$
$$c_d \theta_{d-1}^* = \theta_1 \theta_d^*,$$
$$b_0 \theta_1^* = \theta_1 \theta_0^*.$$

therefore

$$Av_1 = \theta_1 v_1$$

$v_1 \neq 0$ since $d \geq 1$ and the θ_i^* are mut dist.

Note that $$v_1 = A^* v_0$$

claim follows. ✓

Define $$v_d = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{pmatrix} \in V$$

claim 3 $\quad v_d$ is a basis for V_d

pf cl 3 $\quad$ By claim 1, and $\theta_d = -\theta_0$ we find

$$Av_d = \theta_d v_d.$$

Define $$v_{d-1} = \begin{pmatrix} \theta_0^* \\ -\theta_1^* \\ \theta_2^* \\ -\theta_3^* \\ \vdots \end{pmatrix} \in V$$

claim 4 $\quad v_{d-1}$ is a basis for V_{d-1}. Moreover

$$A^* v_d = v_{d-1}$$

pf cl $\quad$ By cl 2 and since $\theta_{d-1} = -\theta_1$

$$Av_{d-1} = \theta_{d-1} v_{d-1}.$$

Note that $A^* v_d = v_{d-1}$ ✓

claim 5

$$A^* V_i \subseteq V_{i-1} + V_{i+1} \qquad 1 \le i \le d-1$$

pf of For $v \in V_i$,

$$0 = \left(A^2 A^* - \beta A A^* A + A^* A^2 - \rho A^* \right) v$$

$$= \left(A^2 - \beta \theta_i A + \theta_i^2 - \rho \right) A^* v$$

$$= \left(A - \theta_{i-1} I \right)\left(A - \theta_{i+1} I \right) A^* v$$

Now

$$(A - \theta_{i+1} I)\, A^* v \in V_{i-1},$$

$$(A - \theta_{i-1} I)\, A^* v \in V_{i+1}$$

So now

$$A^* v = \frac{(A - \theta_{i-1} I) A^* v - (A - \theta_{i+1} I) A^* v}{\theta_{i+1} - \theta_{i-1}}$$

$$\in V_{i-1} + V_{i+1} \qquad \checkmark$$

claim 6 $\quad \dim V_i = 1 \quad 0 \le i \le d$

<u>pf cl</u> Define

$$W = \sum_{i=0}^{d} V_i$$

$W \neq 0$ by claim 1.

By construction

$$AW \subseteq W.$$

By claims 2, 4, 5

$$A^* W \subseteq W$$

Now $W = V$ since V is irred as a module for A, A^* ✓

Result follows since $\dim V = d+1$

By claim 6: A is diagonalizable.

For $0 \le i \le d$ θ_i is an eigenvalue of A, and V_i is corresp eigenspace. Let E_i denote the prim idempotent of A for θ_i. We have shown that

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

is a totally bip TD system with eigenvalue sequence $\{\theta_i\}_{i=0}^{d}$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^{d}$.

By L18 Φ is unique up to iso of TD systems. $\square$

Until further notice $\mathbb{F}$ is alg closed

Motivation Consider the relations

$$A^2A^* - \beta AA^*A + A^*A^2 = \rho A^*,$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} = \rho^* A$$

$\beta = q + q^{-1}$

We will consider the $\mathbb{F}$-algebra defined by generators A, A^* subject to these relations.

First we adjust notation:

Write $B = A^*$

Replace $q \to q^2$ so $\beta = q^2 + q^{-2}$

Assume $q^4 \neq 1$

Multiplying A by a nonzero scalar, WLOG

$$\rho = -(q^2 - q^{-2})^2$$

Similarly WLOG

$$\rho^* = -(q^2 - q^{-2})^2$$

Now LEM 37 (case I) becomes

$$\theta_i = \iota^0 q^{2i-d} + \iota^{0-1} q^{d-2i} \qquad 0 \leq i \leq d$$

$$\theta_i^* = \iota^0 q^{2i-d} + \iota^{0-1} q^{d-2i}$$

$\iota^{0\,2} = -1$

DEF 47 Let Δ_q denote the $\mathbb{F}$-algebra defined by generators A, B subject to the relations

$$A^2B - (q^2+q^{-2})ABA + BA^2 = -(q^2-q^{-2})^2 B,$$

$$B^2A - (q^2+q^{-2})BAB + AB^2 = -(q^2-q^{-2})^2 A.$$

— o —

We now give another presentation of Δ_q

LEM 48 Δ_q has a presentation by generators A, B, C and relations

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = 0, \qquad (1)$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = 0, \qquad (2)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = 0. \qquad (3)$$

pf In (1), (2) elim C using (3) to recover the relations in Def 47. $\square$

<u>Aside</u> Let Δ_q denote the $\mathbb{F}$-algebra defined by gens and rels as follows.

The gens are A, B, C.

The rels assert that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \quad \text{is central}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} \quad \text{"}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \quad \text{"}$$

Δ_q is the <u>universal Askey-Wilson algebra</u>.

$\exists$ surjective $\mathbb{F}$-alg hom $\Delta_q \to \blacktriangle_q$ that sends $A \to A$, $B \to B$, $C \to C$.

Recall the group $PSL_2(\mathbb{Z})$

$SL_2(\mathbb{Z})$ = group of all 2×2 matrices $/ \mathbb{Z}$ that have det 1

Center of $SL_2(\mathbb{Z})$ is $\{\pm I\}$

$$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\}$$

Fact: $PSL_2(\mathbb{Z})$ has a presentation

$$\{\sigma, \rho \mid \sigma^2 = 1,\ \rho^3 = 1\}$$

Thm 49 The group $PSL_2(\mathbb{Z})$ acts on $\blacktriangle_q$ as a group of automorphisms as follows:

u	A	B	C
$\rho(u)$	B	C	A
$\sigma(u)$	B	A	$C + \dfrac{AB-BA}{q-q^{-1}}$

pf By LEM 49 $\exists$ aut P of $\blacktriangle_q$ that sends $A \to B \to C \to A$. Obs $P^3 = I$.

By Def 47 $\exists$ aut S of $\blacktriangle_q$ that sends $A \leftrightarrow B$. Obs

$$C - S(C)$$

$$= -\frac{qAB - q^{-1}BA}{q^2-q^{-2}} + \frac{qBA - q^{-1}AB}{q^2-q^{-2}}$$

$$= -\frac{AB-BA}{q-q^{-1}}$$

Result follows. $\square$

Aside (ex) • the following is a basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i B^j C^k \qquad i,j,k \geq 0$$

this can be shown using L48 and the Bergman Diamond Lemma.

• The elements

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2$$
$$qBCA + q^2B^2 + q^{-2}C^2 + q^2A^2$$
$$qCAB + q^2C^2 + q^{-2}A^2 + q^2B^2,$$

$$q^{-1}CBA + q^{-2}A^2 + q^2B^2 + q^{-2}C^2$$
$$q^{-1}ACB + q^{-2}B^2 + q^2C^2 + q^{-2}A^2$$
$$q^{-1}BAC + q^{-2}C^2 + q^2A^2 + q^{-2}B^2$$

coincide. Call it Ω.

- Ω is central in Δ_q
- Ω generates the center of Δ_q provided q is not a root of 1.
- Ω is fixed by each element of $PSL_2(\mathbb{Z})$

— o —

Next goal: how is Δ_q related to the quantum group $U_q(\mathfrak{sl}_2)$.

Def 50 The $\mathbb{F}$-algebra $U_q = U_q\mathfrak{sl}_2$ is defined by gens e, f, k, k^{-1} and rels

$$kk^{-1} = k^{-1}k = 1$$

$$ke = q^2 ek \qquad kf = q^{-2} fk$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

"Chevalley presentation"

— o —

We now recall the equitable presentation for $U_q\mathfrak{sl}_2$

Prop 51 The algebra U_q is isomorphic to the $\mathbb{F}$-algebra defined by gens x, y, y^{-1}, z and rels

$$yy^{-1} = y^{-1}y = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1$$

"equitable presentation"

An iso with the presentation in Def 50 is

$$y^{\pm 1} \to k^{\pm 1}$$

$$z \to k^{-1} + f(q - q^{-1})$$

$$x \to k^{-1} - ek^{-1}q^{-1}(q - q^{-1})$$

The inverse iso sends

$$k^{\pm 1} \to y^{\pm 1}$$

$$f \to (z - y^{-1})(q - q^{-1})^{-1}$$

$$e \to (1 - xy)\, q\,(q - q^{-1})^{-1}$$

pf One checks each map above is an $\mathbb{F}$-alg hom, and that they are inverses. $\square$

Prop 52 $\exists$ $\mathbb{F}$-alg hom $\Delta_q \to U_q$ that sends

$$A \to i(x-y) + \frac{xy-yx}{q-q^{-1}} \qquad (1)$$

$$B \to i(y-z) + \frac{yz-zy}{q-q^{-1}} \qquad i^2=-1 \qquad (2)$$

$$C \to i(z-x) + \frac{zx-xz}{q-q^{-1}} \qquad (3)$$

pf Let A', B', C' denote the RHS in (1)–(3).

One checks A', B', C' sat the rels (1)–(3) in Lem 48.

$\square$