Lecture 18 Monday Oct 14

We continue to discuss our totally bipartite TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V. Recall Φ is a Leonard system.

In Lem 36 we saw

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \qquad 1 \le i \le d-1 \qquad (R)$$

$$\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = 0 \qquad (R^*)$$

Also by LEM 17

$$\frac{\theta_1}{\theta_0} = \frac{\theta_{d-1}}{\theta_d} = \frac{\theta^*_1}{\theta^*_0} = \frac{\theta^*_{d-1}}{\theta^*_d} \qquad \star$$

We now solve the above equations

Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$

LEM 37 At least one of the following cases I – III hold (and exactly one if $d \geq 3$)

Case I

$\theta_i = h(q^i - q^{d-i})$ $\quad 0 \leq i \leq d$

$\theta_i^* = h^*(q^i - q^{d-i})$

$q, h, h^* \in \overline{\mathbb{F}}$

$q \neq 0, \ q \neq 1, \ q \neq -1$ $\quad h \neq 0, \ h^* \neq 0$

Case II

$\theta_i = h(d-2i)$ $\quad 0 \leq i \leq d$

$\theta_i^* = h^*(d-2i)$

$h, h^* \in \mathbb{F}$

$h \neq 0, \ h^* \neq 0$

Case III (d even only)

$\theta_i = h(d-2i)(-1)^i$ $\quad 0 \leq i \leq d$

$\theta_i^* = h^*(d-2i)(-1)^i$

$h, h^* \in \mathbb{F}$

$h \neq 0, \ h^* \neq 0$

pf Consider the scalar β from Prop 35.

Case I $\beta \neq \pm 2$

$\exists \; 0 \neq q \in \overline{\mathbb{F}}$ such that $\beta = q + q^{-1}$

obs $q \neq \pm 1$

Solve the rec (R)

To find $\exists \; a, b \in \overline{\mathbb{F}}$ such that

$$\theta_i = a q^i + b q^{d-i} \qquad 0 \le i \le d$$

Show $a = -b$:

By ☆

$$0 = \theta_1 \theta_d - \theta_0 \theta_{d-1}$$

$$= (aq + bq^{d-1})(aq^d + b) - (a + bq^d)(aq^{d-1} + bq)$$

$$= (a^2 - b^2) q^d (q - q^{-1})$$

So $a^2 = b^2$

$a \neq b$ else $\theta_0 = \theta_d$

So $a = -b$

$a \neq 0$

Define $h = a$

Get
$$\theta_i = h(q^i - q^{d-i}) \qquad 0 \le i \le d$$

Similarly $\exists\ 0 \neq h^* \in \overline{\mathbb{F}}$ such that
$$\theta_i^* = h^*(q^i - q^{d-i}) \qquad 0 \le i \le d$$

Case II $\beta = 2$

Solve the rec (R) to find $\exists\ a, b \in \mathbb{F}$ s.t.
$$\theta_i = a + bi \qquad 0 \le i \le d$$

By ☆
$$\begin{aligned} 0 &= \theta_1\theta_d - \theta_0\theta_{d-1} \\ &= (a+b)(a+bd) - (a)(a+b(d-1)) \\ &= b(2a+bd) \end{aligned}$$

$b \neq 0$ else $\theta_0 = \theta_1$

So $a = \frac{-bd}{2}$

define $\quad h = -b/2$

Get
$$\theta_i = h(d-2i) \qquad 0 \le i \le d$$

Similarly $\quad \exists\ 0 \ne h^* \in \mathbb{F}$ s.t.

$$\theta_i^* = h^*(d-2i) \qquad 0 \le i \le d$$

<u>Case III</u> $\quad \beta = -2$

For $d \le 2$ $\quad \beta$ is not uniquely det; choice of β is arb.

So in this case, WLOG $d \ge 2$

Solve the rec (R) to find $\exists\ a, b \in \mathbb{F}$ such that

$$\theta_i = (a + bi)(-1)^i \qquad 0 \le i \le d$$

By ☆,

$$0 = \theta_1 \theta_d - \theta_0 \theta_{d-1}$$
$$= (-1)^d b(2a + bd)$$

$b \ne 0$ else $\theta_0 = \theta_2$

So $\quad a = -\frac{d}{2} b$

Define $\quad h = -b/2$

Get

$$\theta_i = h(d-2i)(-1)^i \qquad 0 \le i \le d$$

Similarly $\exists$ $0 \ne h^* \in \mathbb{F}$ s.t.

$$\theta_i^* = h^*(d-2i)(-1)^i \qquad 0 \le i \le d$$

Note d is even, else $\theta_0 = \theta_d$. $\square$

LEM 38 For $0 \le i \le d$

$$\theta_i + \theta_{d-i} = 0,$$

$$\theta_i^* + \theta_{d-i}^* = 0.$$

$\square$

pf Use the formulae in LEM 37

LEM 39 The integer n in LEM 34 is equal to d

pf Recall $\theta_n^* = -\theta_0^*$ $\square$

LEM 40 For $0 \le i, j \le d$

Case I

$$\theta_i - \theta_j = h(q^i - q^j)(1 + q^{d-i-j})$$

$$\theta^*_i - \theta^*_j = h^*(q^i - q^j)(1 + q^{d-i-j})$$

Case II

$$\theta_i - \theta_j = 2h(j - i)$$

$$\theta^*_i - \theta^*_j = 2h^*(j - i)$$

Case III

$\theta_i - \theta_j =$

	j even	j odd
i even	$2h(j-i)$	$2h(d-i-j)$
i odd	$2h(i+j-d)$	$2h(i-j)$

$\theta^*_i - \theta^*_j =$

	j even	j odd
i even	$2h^*(j-i)$	$2h^*(d-i-j)$
i odd	$2h^*(i+j-d)$	$2h^*(i-j)$

pf Use LEM 37

LEM 41

Case I

$q^i \neq 1 \qquad 1 \le i \le d$

$q^i \neq -1 \qquad 1 \le i \le d-1$

Case II

char $\mathbb{F} = 0$ or char $\mathbb{F} > d$

Case III

char $\mathbb{F} = 0$ or char $\mathbb{F} > d$.

pf Recall

$\{\theta_i\}_{i=0}^{d}$ are not distinct

$\{\theta_i^*\}_{i=0}^{d}$ ...

Now use L40. $\square$

LEM 42 The scalars β, ρ, ρ^* from Prop 35 satisfy

Case I

$$\beta = q + q^{-1}$$

$$\rho = h^2 q^d (q - q^{-1})^2$$

$$\rho^* = h^* q^d (q - q^{-1})^2$$

Case II

$$\beta = 2$$

$$\rho = 4h^2$$

$$\rho^* = 4h^{*2}$$

Case III

$$\beta = -2$$

$$\rho = 4h^2$$

$$\rho^* = 4h^{*2}$$

pf Recall by L36

$$\rho = \theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 \qquad 1 \le i \le d$$

$$\rho^* = \theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2}$$

Now use L37. $\square$

LEM 43 We have

Case I

$$c_i = h \frac{1 - q^{2i}}{1 + q^{2i-d}} \qquad 1 \le i \le d-1$$

$$c_d = h(1 - q^d)$$

$$b_i = h \frac{1 - q^{2d-2i}}{1 + q^{d-2i}} \qquad 1 \le i \le d-1$$

$$b_0 = h(1 - q^d)$$

Case II

$$c_i = hi \qquad 0 \le i \le d$$

$$b_i = h(d-i)$$

Case III

$$c_i = hi \qquad 0 \le i \le d$$

$$b_i = h(d-i)$$

To get c_i^*, b_i^* replace h by h^*

pf Use LEM 18 and LEM 37. $\square$

LEM 44 For $0 \le i \le d$,

$$b_i = c_{d-i} \qquad b_i^* = c_{d-i}^*$$

pf L43 $\square$

Recall the scalar

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$

LEM 45 For $0 \le i \le d$

$$k_i = k_{d-i} \qquad k_i^* = k_{d-i}^*$$

pf use L44 $\square$

Thm 46 Given integer $d \geq 1$. Given scalars

$$\theta_i, \quad \theta_i^* \qquad 0 \leq i \leq d$$

in $\mathbb{F}$. Then $\exists$ totally bipartite TD system Φ over $\mathbb{F}$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$ iff the following (i)–(iii) hold:

(i) $\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^*$ if $i \neq j$ $\quad (0 \leq i, j \leq d)$

(ii) For $0 \leq i \leq d$,

$$\theta_i + \theta_{d-i} = 0, \qquad \theta_i^* + \theta_{d-i}^* = 0$$

(iii) $\exists \; \beta \in \mathbb{F}$ s.t.

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0, \qquad 1 \leq i \leq d-1$$

$$\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0$$

Suppose (i)–(iii) hold. Then Φ is unique up to iso of TD systems.