LECTURE 17 Friday Oct 11

We continue to discuss our totally bipartite TD system $\Phi = (A, \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V

We saw earlier that Φ is a Leonard system.

LEM 23 For $\beta, \rho \in \mathbb{F}$ TFAE:

(i) $\theta_{i-1}^2 - \beta\,\theta_{i-1}\theta_i + \theta_i^2 = \rho \qquad 1 \le i \le d$

(ii) $A^2A^* - \beta AA^*A + A^*A^2 = \rho A^*$

pf Define $C = A^2A^* - \beta AA^*A + A^*A^2 - \rho A^*$

View $C = ICI \qquad I = \sum_{i=0}^d E_i$

$$= \sum_{0 \le i,j \le d} E_i C E_j$$

So $C = 0$ iff $E_i C E_j = 0$ for $0 \le i,j \le d$.

For $0 \le i,j \le d$

$$E_i C E_j = E_i A^* E_j \left(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \rho \right)$$

Also $E_i A^* E_j = 0$ iff $|i-j| \neq 1$

Result follows. $\square$

Define M = subalgebra of $\mathrm{End}(V)$ gen by A

" M^* = ... A^*

Define

$$MA^*M = \mathrm{Span}\{uA^*v \mid u, v \in M\}$$

LEM 24 The following is a basis for the $\mathbb{F}$-vector space MA^*M:

$$E_i A^* E_j \qquad |i-j| = 1 \qquad (0 \le i, j \le d) \qquad (*)$$

Moreover the dimension of MA^*M is $2d$

pf the elements $(*)$ span MA^*M since $E_0, \ldots, E_d$ span M and $E_i A^* E_j = 0$ if $|i-j| \neq 1$.

Using $E_r E_s = \delta_{rs} E_r$ $(0 \le r, s \le d)$ one checks that $(*)$ are lin indep. So $(*)$ is a basis for $(*)$.

This basis has $2d$ elements.

$\square$

LEM 25

(i) $E_0 A^* = E_0 A^* E_1$

(ii) $E_i A^* = E_i A^* E_{i-1} + E_i A^* E_{i+1}$ $(1 \le i \le d-1)$

(iii) $E_d A^* = E_d A^* E_{d-1}$

pf Fa $0 \le i \le d$ view

$E_i A^* = E_i A^* I$ $I = \sum_{j=0}^{d} E_j$

Recall

$E_i A^* E_j = 0$ if $|i-j| \ne 1$

$\square$

LEM 26

(i) $$A^* E_0 = E_1 A^* E_0$$

(ii) $$A^* E_i = E_{i-1} A^* E_i + E_{i+1} A^* E_i \qquad (1 \le i \le d-1)$$

(iii) $$A^* E_d = E_{d-1} A^* E_d$$

pf Apply the antiautomorphism $\dagger$ to everything in LEM 25. $\square$

LEM 27 For $1 \le i \le d$,

$$E_{i-1} A^* E_i = E_{i-1} A^* - A^* E_{i-2} + E_{i-3} A^* - A^* E_{i-4} + \cdots$$

$$E_i A^* E_{i-1} = A^* E_{i-1} - E_{i-2} A^* + A^* E_{i-3} - E_{i-4} A^* + \cdots$$

Moreover

$$\sum_{\substack{0 \le i \le d \\ i \text{ even}}} E_i A^* = \sum_{\substack{0 \le i \le d \\ i \text{ odd}}} A^* E_i ,$$

$$\sum_{\substack{0 \le i \le d \\ i \text{ odd}}} E_i A^* = \sum_{\substack{0 \le i \le d \\ i \text{ even}}} A^* E_i .$$

pf Solve the equations in LEM 25, 26. $\square$

LEM 28 The following is a basis for the $\mathbb{F}$-vector space MA^*M:

$$E_iA^*, \quad A^*E_i \qquad 0 \le i \le d \qquad (*)$$

pf By construction

$$\mathrm{Span}(*) \subseteq MA^*M$$

By LEM 24 and LEM 27

$$MA^*M \subseteq \mathrm{Span}(*)$$

So $$MA^*M = \mathrm{Span}(*).$$

There are $2d$ elements in $(*)$. These elements are lin indep since MA^*M has dimension $2d$.

$\square$

COR 29

$$MA^*M = MA^* + A^*M$$

pf $\subseteq$ By LEM 28

$\supseteq$ Clear

$\square$

Define

$$(MA^*M)^{sym} = \{ u \in MA^*M \mid u^t = u \}$$

"symmetric part of MA^*M"

LEM 30 The following is a basis for the $\mathbb{F}$-vector space $(MA^*M)^{sym}$:

$$E_i A^* E_{i-1} + E_{i-1} A^* E_i \qquad 1 \le i \le d$$

Moreover the dimension of $(MA^*M)^{sym}$ is d.

pf By LEM 24 and since

$$(E_r A^* E_s)^t = E_s A^* E_r \qquad 0 \le r, s \le d.$$

$\square$

LEM 31 The following is a basis for the $\mathbb{F}$-vector space $(MA^*M)^{sym}$:

$$E_iA^* + A^*E_i \qquad 0 \le i \le d-1$$

pf By LEM 28 and since

$$(E_rA^*)^\dagger = A^*E_r \qquad 0 \le r \le d$$

$$(A^*E_r)^\dagger = E_rA^* \qquad \text{"}$$

$\square$

LEM 32

$$(MA^*M)^{sym} = \{ uA^* + A^*u \mid u \in M \}$$

pf $\subseteq$: By LEM 31

$\supseteq$: $\forall u \in M$

$$(A^*u)^+ = uA^*$$

$$(uA^*)^+ = A^*u$$

So $+$ leaves

$$uA^* + A^*u$$

invariant. $\square$

LEM 33 The $\mathbb{F}$-vector space

$(MA^*M)^{sym}$ is spanned by

$$A^*, \quad AA^*+A^*A, \quad A^2A^*+A^*A^2, \ldots, A^dA^*+A^*A^d. \qquad (*)$$

Pf M has basis $\{A^i\}_{i=0}^d$ so by LEM 32

$$(MA^*M)^{sym} = \text{Span}\{A^iA^*+A^*A^i\}_{i=0}^d$$

$$\subseteq \text{Span}(*)$$

Also

$$\text{Span}(*) \subseteq (MA^*M)^{sym}$$

by LEM 32 and the def of $(MA^*M)^{sym}$ $\square$

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There are $d+1$ vectors in $(*)$ and $(MA^*M)^{sym}$ has dimension d.

So the vectors $(*)$ are lin dependent.

We now find this dependency.

LEM 34

(i) char $\mathbb{F} \neq 2$

(ii) $\exists$ integer n $(1 \le n \le d)$ such that $\theta_n^* = -\theta_0^*$

(iii) $A^n A^* + A^* A^n$ is in the span of

$$A^*, \quad AA^* + A^*A, \quad A^2A^* + A^*A^2, \ldots, \quad A^{n-1}A^* + A^*A^{n-1}.$$

pf In LEM 33 the vectors $(*)$ are lin dep.

So $\exists$ scalars $\{\alpha_i\}_{i=0}^d$ in $\mathbb{F}$, not all 0, such that

$$\alpha_0 A^* + \sum_{i=1}^{d} \alpha_i \left(A^i A^* + A^* A^i \right) = 0$$

$A^* \neq 0$ Since Φ is non trivial.

So $\exists\, i$ $(1 \le i \le n)$ such that $\alpha_i \neq 0$. Define

$$n = \max \{ i \mid 1 \le i \le d, \ \alpha_i \neq 0 \}$$

So
$$\alpha_0 A^* + \sum_{i=1}^{n} \alpha_i \left(A^i A^* + A^* A^i \right) = 0, \qquad \alpha_n \neq 0$$

Now
$$0 = E_0^* \left(\alpha_0 A^* + \sum_{i=1}^{n} \alpha_i \left(A^i A^* + A^* A^i \right) \right) E_n^*$$

$$\left[E_0^* A^i E_n^* = 0 \text{ if } i < n \right]$$

$$= \underbrace{E_0^* A^n E_n^*}_{\neq 0} \underbrace{\alpha_n}_{\neq 0} \left(\theta_n^* + \theta_0^* \right)$$

So $\theta_n^* + \theta_0^* = 0$

So $\theta_n^* = -\theta_0^*$

Now char $\mathbb{F} \neq 2$ else $\theta_n^* = \theta_0^*$ cont. Result follows. $\square$

Referring to LEM 34, we will show $n=d$.

Prop 35 $\exists$ scalars β, ρ, ρ^* in $\mathbb{F}$ such that

$$A^2A^* - \beta AA^*A + A^*A^2 = \rho A^* \quad (1)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} = \rho^* A \quad (2)$$

The sequence β, ρ, ρ^* is unique if $d \geq 3$.

pf Observe

$$AA^*A \in (MA^*M)^{sym}$$

By LEM 33 and LEM 34 $\exists$ $\{\alpha_i\}_{i=0}^{d}$ in $\mathbb{F}$

with $\alpha_n = 0$ such that

$$AA^*A = \alpha_0 A^* + \sum_{i=1}^{d} \alpha_i \left(A^iA^* + A^*A^i\right)$$

claim 1 $\quad \alpha_i = 0 \quad 3 \le i \le d$

pf cl $\quad$ Suppose not. Then $d \ge 3$.

Define $\quad t = \max\{ i \mid 3 \le i \le d, \ \alpha_i \ne 0 \}$

Then $\quad t \ne n$

and

$$A A^* A = \alpha_0 A^* + \sum_{i=1}^{t} \alpha_i (A^i A^* + A^* A^i)$$

So

$$0 = E_0^* \left(\alpha_0 A^* + \sum_{i=1}^{t} \alpha_i (A^i A^* + A^* A^i) - A A^* A \right) E_t^*$$

$$= \underbrace{E_0^* A^t E_t^*}_{\ne 0} \underbrace{\alpha_t}_{\ne 0} \underbrace{(\theta_t^* + \theta_0^*)}_{\ne 0 \text{ by } t \ne n} \qquad \text{by L20}$$

$\ne 0$

cont. $\qquad$ claim proved.

<u>Claim 2</u> $\exists\, \beta \in \mathbb{F}$ such that

$$\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = 0 \qquad 1 \le i \le d-1.$$

<u>pf of</u> Assume $d \ge 2$; otherwise there is nothing to show.

By claim 1,

$$0 = \alpha_0 A^* + \alpha_1(AA^* + A^*A) + \alpha_2(A^2A^* + A^*A^2) - AA^*A.$$

So for $1 \le i \le d-1$,

$$0 = E^*_{i-1}\left(\alpha_0 A^* + \alpha_1(AA^* + A^*A) + \alpha_2(A^2A^* + A^*A^2) - AA^*A\right)E^*_{i+1}$$

[use LEM 20]

$$= \underbrace{E^*_{i-1}A^2E^*_{i+1}}_{\ne 0}\left(\alpha_2\theta^*_{i+1} + \alpha_2\theta^*_{i-1} - \theta^*_i\right)$$

So

$$\alpha_2(\theta^*_{i-1} + \theta^*_{i+1}) = \theta^*_i \qquad 1 \le i \le d-1$$

Assume $\alpha_2 \ne 0$. Put $\beta = \frac{1}{\alpha_2}$ and we are done.

Assume $\alpha_2 = 0$. Then

$$\theta^*_i = 0 \qquad 1 \le i \le d-1$$

So $d = 2$ and $\theta^*_1 = 0$

In this case $n = 2$ and

$$\theta^*_2 = -\theta^*_0$$

Let $\beta \in \mathbb{F}$ be arbitrary. Then $\theta^*_0 - \beta\theta^*_1 + \theta^*_2 = 0$ and we are done. claim proved.

By claim 2 and LEM 21,

$\exists\ \rho^* \in \mathbb{F}$ such that

$$\theta_{i-1}^{*2} - \beta\theta_{i-1}^* \theta_i^* + \theta_i^{*2} = \rho^* \qquad 1 \le i \le d$$

Now by LEM 23 (applied to Φ^*)

$$A^{*2}A - \beta A^*AA^* + AA^{*2} = \rho^* A$$

Now for $1 \le i \le d-1$

$$0 = E_{i-1}\left(A^{*2}A - \beta A^*AA^* + AA^{*2} - \rho^* A\right)E_{i+1}$$

[apply L20 to Φ^*]

$$= \underbrace{E_{i-1}A^{*2}E_{i+1}}_{\neq 0}\left(\theta_{i+1} - \beta\theta_i + \theta_{i-1}\right)$$

So

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \qquad 1 \le i \le d-1$$

By LEM 21 $\exists\ \rho \in \mathbb{F}$ such that

$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 = \rho \qquad 1 \le i \le d$$

Now by LEM 23

$$A^2A^* - \beta AA^*A + A^*A^2 = \rho A^*$$

Now assume $d \geq 3$.

We show that the sequence β, ρ, ρ^* is unique.

By LEM 23

$$\theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 = \rho \qquad 1 \leq i \leq d \qquad (*)$$

By LEM 22

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = 0 \qquad 1 \leq i \leq d-1 \qquad (**)$$

The scalar β is determined by $(**)$.
The scalar ρ is determined by $(*)$.
The scalar ρ^* is similarly determined. $\square$

We record some results from the proof of Prop 35.

LEM 36 Given $\beta, \rho, \rho^* \in \mathbb{F}$ that satisfy Prop 35. Then

(i) $\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 = \rho \qquad 1 \le i \le d$

(ii) $\theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} = \rho^* \qquad 1 \le i \le d$

(iii) $\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \qquad 1 \le i \le d-1$

(iv) $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0 \qquad 1 \le i \le d-1$

$\square$