LECTURE 16 Wednesday Oct 9

Next goal: classify up to iso the totally bip TD systems.

$\mathbb{F}$ = arb field

V = vector space over $\mathbb{F}$ with finite pos dim

Fix a TD system on V:

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$$

that is totally bipartite

To avoid trivialities assume Φ is nontrivial.
So $d \geq 1$, $\delta \geq 1$.

Define

$$R = \sum_{i=1}^{\delta} E_i^* A E_{i-1}^* \qquad \text{"Raising map"}$$

$$L = \sum_{i=1}^{\delta} E_{i-1}^* A E_i^* \qquad \text{"Lowering map"}$$

So

$$A = R + L$$

Obs

$$R\, E_i^* V \subseteq E_{i+1}^* V \qquad (0 \le i \le \delta - 1),$$

$$R\, E_\delta^* V = 0$$

$$L\, E_i^* V \subseteq E_{i-1}^* V \qquad (1 \le i \le \delta),$$

$$L\, E_0^* V = 0.$$

Fix $0 \neq v \in E_0 V$

Define

$$v_i = E_i^* v \qquad 0 \le i \le \delta$$

So

$$v_i \in E_i^* V \qquad 0 \le i \le \delta$$

and

$$v = \sum_{i=0}^{\delta} v_i.$$

Obs $\quad Lv_0 = 0, \qquad Rv_\delta = 0 \qquad (*)$

LEM 12

(i) $\theta_0 v_0 = L v_1$

(ii) $\theta_0 v_i = R v_{i-1} + L v_{i+1}$ $(1 \le i \le \delta-1)$

(iii) $\theta_0 v_\delta = R v_{\delta-1}$

pf For $0 \le i \le \delta$ define

$$\tilde{v}_i = R v_{i-1} + L v_{i+1} - \theta_0 v_i$$

where $v_{-1} = 0$, $v_{\delta+1} = 0$.

Obs $\tilde{v}_i \in E_i^* V$.

Show $\tilde{v}_i = 0$.

Since $v \in E_0 V$,

$$A v = \theta_0 v$$

So

$$0 = \underbrace{(A - \theta_0 I)}_{R+L} \underbrace{v}_{\sum_{i=0}^{\delta} v_i}$$

$$= \sum_{i=0}^{\delta} \tilde{v}_i$$

the sum $V = \sum_{i=0}^{\delta} E_i^* V$ is direct so

$$\tilde{v}_i = 0 \qquad 0 \le i \le \delta$$

$\square$

LEM 13

(i) $\theta_1 \theta_0^* v_0 = \theta_1^* L v_1$

(ii) $\theta_1 \theta_i^* v_i = \theta_{i-1}^* R v_{i-1} + \theta_{i+1}^* L v_{i+1} \quad (1 \le i \le \delta-1)$

(iii) $\theta_1 \theta_\delta^* v_\delta = \theta_{\delta-1}^* R v_{\delta-1}$

pf Recall $v \in E_0 V$ so

$$A^* v \in A^* E_0 V \subseteq E_1 V$$

So

$$0 = \underbrace{(A - \theta_1 I)}_{= R+L} \underbrace{A^* v}_{= \sum_{i=0}^{\delta} \theta_i^* v_i}$$

Expand and collect terms as in pf of L 12.

$\square$

LEM 14 We have

$$R v_{i-1} = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} v_i \qquad (1 \le i \le \delta-1)$$

$$R v_{\delta-1} = \theta_0 v_\delta$$

$$L v_{i+1} = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i-1}^*}{\theta_{i+1}^* - \theta_{i-1}^*} v_i \qquad (1 \le i \le \delta-1)$$

$$L v_1 = \theta_0 v_0$$

Also

$$(\theta_0 \theta_1^* - \theta_0^* \theta_1) v_0 = 0,$$

$$(\theta_0 \theta_{\delta-1}^* - \theta_\delta^* \theta_1) v_\delta = 0.$$

pf Solve the Linear equations in Lems 12, 13. $\square$

LEM 15

(i) v_i is a basis for $E_i^* V$ $\quad (0 \le i \le \delta)$

(ii) $\{v_i\}_{i=0}^{\delta}$ is a basis for V "Φ- standard basis"

(iii) $\dim V = \delta + 1$

pf (i) Define

$$W = \text{Span}\{v_i\}_{i=0}^{\delta}$$

$$0 \neq v = \sum_{i=0}^{\delta} v_i \in W$$

So $W \neq 0$

By LEM 14 and (*)

$$RW \subseteq W, \qquad LW \subseteq W$$

So

$$AW \subseteq W \qquad A = R + L$$

Obs $A^* v_i = \theta_i^* v_i \qquad 0 \le i \le \delta$

So $A^* W \subseteq W$

Now $W = V$ since V is irred as a module for A, A^*.

Now for $0 \le i \le \delta$, $0 \neq E_i^* V = \mathbb{F} v_i$

So $v_i \neq 0$. (ii), (iii) Clear from (i) $\square$

COR 16 We have $d=\delta$ and

$$\dim E_i V = 1, \qquad \dim E_i^* V = 1 \qquad (0 \le i \le d)$$

In other words, our TD system Φ is a Leonard system.

pf Apply LEM 15 to Φ and Φ^*. $\square$

LEM 17

(i) Each of $\theta_0, \theta_d, \theta_0^*, \theta_d^*$ is nonzero.

(ii) $$\frac{\theta_1}{\theta_0} = \frac{\theta_{d-1}}{\theta_d} = \frac{\theta_1^*}{\theta_0^*} = \frac{\theta_{d-1}^*}{\theta_d^*}$$

pf By the last assertion of LEM 14 and since $V_0 \neq 0$, $V_\delta \neq 0$

$$\theta_0 \theta_1^* = \theta_0^* \theta_1, \qquad (1)$$

$$\theta_0 \theta_{\delta-1}^* = \theta_\delta^* \theta_1 \qquad (2)$$

Applying this to Φ^*,

$$\theta_0^* \theta_{\delta-1} = \theta_\delta \theta_1^* \qquad (3)$$

Suppose $\theta_0 = 0$ then $\theta_1 \neq 0$ then $\theta_0^* = 0$ by (1) and $\theta_\delta^* = 0$ by (2). So $\theta_0^* = \theta_\delta^*$ contr.

Therefore $\theta_0 \neq 0$. Applying this to the relatives of Φ,

$$\theta_d \neq 0, \quad \theta_0^* \neq 0, \quad \theta_d^* \neq 0$$

This gives (i) and (ii) follows from (1)–(3). $\square$

For $X \in \mathrm{End}(V)$ let $X^{\natural}$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that represents X with respect to the Φ-standard basis $\{v_i\}_{i=0}^{d}$ of V

Thus

$$X v_j = \sum_{i=0}^{d} (X^{\natural})_{ij} v_i \qquad (0 \le j \le d)$$

Obs

$$\natural : \begin{array}{ccc} \mathrm{End}(V) & \to & \mathrm{Mat}_{d+1}(\mathbb{F}) \\ X & \mapsto & X^{\natural} \end{array}$$

is $\mathbb{F}$-alg isomorphism.

By construction

$$(E_i^*)^{\natural} = \mathrm{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ \text{coord } i}}{1}, 0, \ldots, 0) \qquad (0 \le i \le d)$$

$$(A^*)^{\natural} = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$$

By constr

$$A^\natural = \begin{pmatrix} 0 & b_0 & & & & \mathbf{0} \\ c_1 & 0 & b_1 & & & \\ & c_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & \ddots & b_{d-1} \\ & & & & c_d & 0 \end{pmatrix}$$

for some $c_i, b_i \in \mathbb{F}$

Obs

$$c_i \neq 0 \qquad 1 \le i \le d,$$

$$b_i \neq 0 \qquad 0 \le i \le d-1,$$

Since V is irred as a module for A, A^*.

Call $\{c_i\}_{i=1}^{d}$, $\{b_i\}_{i=0}^{d-1}$ the <u>intersection numbers</u> of Φ. For notational convenience $c_0 = 0$, $b_d = 0$.

Let $\{c_i^*\}_{i=1}^{d}$, $\{b_i^*\}_{i=0}^{d-1}$ denote the intersection numbers for Φ^*. Call these the <u>dual intersection numbers</u> of Φ.

LEM 18

(i) the intersection numbers of Φ satisfy

$$c_i = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \qquad (1 \le i \le d-1)$$

$$c_d = \theta_0$$

$$b_i = \frac{\theta_1 \theta_i^* - \theta_0 \theta_{i-1}^*}{\theta_{i+1}^* - \theta_{i-1}^*} \qquad (1 \le i \le d-1)$$

$$b_0 = \theta_0$$

(ii) To get the dual intersection numbers $\{c_i^*\}_{i=1}^{d}$, $\{b_i^*\}_{i=0}^{d-1}$ in (i) swap $\theta_j \leftrightarrow \theta_j^*$ for $0 \le j \le d$.

pf (i) Use LEM 14

(ii) Apply (i) to Φ^* $\square$

COR 19 Up to iso, Φ is determined by its eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$.

pf By LEM 18. $\square$

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Next step: show $\exists\ \beta \in \mathbb{F}$ such that

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \qquad (1 \le i \le d-1)$$

$$\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0$$

Comments

By LEM 1, End(V) has a basis

$$A^r E_0^* A^s \qquad 0 \le r, s \le d$$

By Cor 2

$$A, E_0^* \text{ gen } \mathrm{End}(V)$$

By Cor 6 $\quad A, A^* \text{ gen } \mathrm{End}(V)$

From above LEM 1, for $0 \le i, j, r \le d$

$$E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } |i-j| > r \\ \neq 0 & \text{if } |i-j| = r \end{cases}$$

By LEM 4 $\exists$ unique antiaut $\dagger$ of End(V) that fixes each of A, A^*

By LEM 5 $\exists$ sym, nondeg bilinear form $\langle\, , \rangle$ on V such that

$$\langle Bu, v \rangle = \langle u, Bv \rangle \qquad \forall B \in \mathrm{End}(V) \quad \forall u, v \in V$$

LEM 20 Fn $0 \le i,j,r,s \le d$,

$$E_i^* A^r A^* A^s E_j^* = \begin{cases} \theta^*_{j+s} \; E_i^* A^{r+s} E_j^* & \text{if } i-j = r+s \\ \theta^*_{i+r} \; E_i^* A^{r+s} E_j^* & \text{if } j-i = r+s \\ 0 & \text{if } |i-j| > r+s \end{cases}$$

pf Recall

$$A^* = \sum_{l=0}^{d} \theta_l^* E_l^*$$

So

$$E_i^* A^r A^* A^s E_j^* = \sum_{l=0}^{d} \theta_l^* \underbrace{E_i^* A^r E_l^*}_{\substack{\| \\ 0 \text{ if } |l-i|>r}} \overbrace{ A^s E_j^*}^{\substack{0 \text{ if } |l-j|>s \\ \|}}$$

Result follows. □

<u>Aside</u> Let $\{\sigma_i\}_{i=0}^{d}$ denote any sequence of scalars in $\mathbb{F}$

LEM 21 Given $\beta \in \mathbb{F}$. Assume

$$\sigma_{i-1} - \beta \sigma_i + \sigma_{i+1} = 0 \qquad (1 \le i \le d-1)$$

then $\exists\, \rho \in \mathbb{F}$ s.t.

$$\sigma_{i-1}^2 - \beta \sigma_{i-1}\sigma_i + \sigma_i^2 = \rho \qquad 1 \le i \le d.$$

pf Define

$$S_i = \sigma_{i-1}^2 - \beta \sigma_{i-1}\sigma_i + \sigma_i^2 \qquad 1 \le i \le d$$

Show S_i is indep of i for $1 \le i \le d$.

For $1 \le i \le d-1$,

$$S_i - S_{i+1} = (\sigma_{i-1} - \sigma_{i+1})\underbrace{(\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1})}_{= 0}$$

$$= 0$$

$\square$

LEM 22 Given $\beta, \rho \in \mathbb{F}$ Assume

$$\sigma_{i-1}^2 - \beta\sigma_{i-1}\sigma_i + \sigma_i^2 = \rho \qquad (1 \le i \le d)$$

and

$$\sigma_{i-1} \ne \sigma_{i+1} \qquad (1 \le i \le d-1)$$

then

$$\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = 0 \qquad (1 \le i \le d-1)$$

pf We saw

$$S_i - S_{i+1} = (\sigma_{i-1} - \sigma_{i+1})(\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1})$$

for $1 \le i \le d-1$ where

$$S_i = \sigma_{i-1}^2 - \beta\sigma_{i-1}\sigma_i + \sigma_i^2$$

Result follows. $\square$