Lecture 15 Monday Oct 7

Chapter II TD pairs and Leonard pairs

For this chapter the field $\mathbb{F}$ is arbitrary

Fix integer $d \geq 0$

$\mathrm{Mat}_{d+1}(\mathbb{F})$ is the $\mathbb{F}$-algebra of all $d+1 \times d+1$ matrices with entries in $\mathbb{F}$. Index the rows/columns by $0, 1, \ldots, d$.

Given a tridiagonal matrix $A \in \mathrm{Mat}_{d+1}(\mathbb{F})$:

$$A = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}$$

Call A <u>irreducible</u> whenever $c_i b_{i-1} \neq 0 \quad 1 \leq i \leq d$

Until further notice assume A is irreducible

Note that for $0 \le i, j, r \le d$,

$$(A^r)_{ij} = \begin{cases} 0 & \text{if } |i-j| > r \\ \neq 0 & \text{if } |i-j| = r \end{cases}$$

Therefore $\{A^r\}_{r=0}^{d}$ are lin indep

Therefore the min poly of A = the char poly of A

Therefore each eigenspace of A has dim 1

(Caution: possibly A is not diagonalizable)

For $0 \le i \le d$ define $E_i^* \in \mathrm{Mat}_{d+1}(\mathbb{F})$ by

$$E_i^* = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$$

$\uparrow$ coord i

So

$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \le i, j \le d)$$

$$I = \sum_{i=0}^{d} E_i^*$$

Note that for $0 \le i, j, r \le d$

$$E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } |i-j| > r \\ \neq 0 & \text{if } |i-j| = r \end{cases}$$

In particular for $0 \le i, j \le d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1. \end{cases}$$

LEM 1 The following is a basis for the $\mathbb{F}$-vector space $\mathrm{Mat}_{d+1}(\mathbb{F})$:

$$A^r E_0^* A^s \qquad 0 \le r, s \le d$$

pf For $0 \le i, j \le d$ the (i,j)-entry

$$\left(A^r E_0^* A^s\right)_{ij} = (A^r)_{i0} (A^s)_{0j}$$

$$= \begin{cases} 0 & \text{if } i > r \text{ or } j > s \\ \neq 0 & \text{if } i = r \text{ and } j = s \end{cases}$$

Result follows. □

COR 2 The $\mathbb{F}$-algebra $\mathrm{Mat}_{d+1}(\mathbb{F})$ is generated by A, E_0^*.

pf By LEM 1 □

Define

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad 0 \le i \le d$$

So

$$k_0 = 1$$

$$k_i \ne 0 \qquad 0 \le i \le d.$$

Define $K \in \mathrm{Mat}_{d+1}(\mathbb{F})$ by
$$K = \mathrm{diag}(k_0, k_1, \ldots k_d)$$

LEM 3 We have

$$A^t K = K A$$

pf routine $\square$

For any $\mathbb{F}$-algebra $\mathcal{A}$, by an <u>anti automorphism</u> of $\mathcal{A}$ we mean an $\mathbb{F}$-linear bijection $\sigma : \mathcal{A} \to \mathcal{A}$ that sends

$$(ab)^\sigma = b^\sigma a^\sigma \quad \forall\, a,b \in \mathcal{A}$$

LEM 4. $\exists$ unique antiaut $\dagger$ of $\mathrm{Mat}_{d+1}(\mathbb{F})$ that fixes A and each of $E_0^*, E_1^*, \ldots, E_d^*$. Moreover $\dagger^2 = 1$

pf Existence: The map

$$\dagger : \begin{array}{ccc} \mathrm{Mat}_{d+1}(\mathbb{F}) & \longrightarrow & \mathrm{Mat}_{d+1}(\mathbb{F}) \\ B & \longrightarrow & K^{-1}B^t K \end{array}$$

is an antiaut of $\mathrm{Mat}_{d+1}(\mathbb{F})$ that meets the requirements. Note that $\dagger^2 = 1$.

Uniqueness: let σ denote any antiaut of $\mathrm{Mat}_{d+1}(\mathbb{F})$ that meets the requirements.

The composition $\sigma\dagger$ is an aut of $\mathrm{Mat}_{d+1}(\mathbb{F})$ that fixes $A, E_0^*, \ldots, E_d^*$. Now $\sigma\dagger = 1$ by LEM 2. So $\sigma = \dagger$

$\square$

Consider the $\mathbb{F}$-vector space

$$V = \mathbb{F}^{d+1} \qquad \text{(column vectors)}$$

Index the rows $0, 1, \ldots, d$.

V is a module for $\mathrm{Mat}_{d+1}(\mathbb{F})$

LEM 5 (i) $\exists$ nonzero bilinear form

$$\langle\, , \rangle \qquad V \times V \to \mathbb{F}$$

such that

$$\langle Bu, v \rangle = \langle u, B^t v \rangle \qquad \begin{array}{l} \forall u, v \in V \\ \forall B \in \mathrm{Mat}_{d+1}(\mathbb{F}) \end{array}$$

(ii) $\langle\, , \rangle$ is symmetric and nondegenerate.

(iii) $\langle\, , \rangle$ is unique up to mult by a nonzero scalar in $\mathbb{F}$

pf (i) Define

$$\begin{array}{rcl} V \times V & \to & \mathbb{F} \\ \langle\, , \rangle \qquad u \quad v & \to & u^t K v \end{array}$$

By LEM 4

$$\langle Bu, v \rangle = u^t B^t K v = u^t K K^{-1} B^t K v = u^t K B^+ v$$
$$= \langle u, B^+ v \rangle$$

(ii) clear

(iii) Routine. $\square$

Note that for $u, v \in V$

$$\langle Au, v \rangle = \langle u, Av \rangle$$

$$\langle E_i^* u, v \rangle = \langle u, E_i^* v \rangle \qquad 0 \le i \le d$$

Let A^* denote a diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$.

Write $\qquad A^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*) \qquad \theta_i^* \in \mathbb{F}$

So $\qquad A^* = \sum_{i=0}^{d} \theta_i^* E_i^*$

Obs $\quad A^{*t} = A^*$ so

$$\langle A^* u, v \rangle = \langle u, A^* v \rangle \qquad \forall u, v \in V$$

LEM 6 $\quad$ Assume $\theta_0^* \ne \theta_i^*$ $\quad 1 \le i \le d$. Then

(i) $\quad E_0^* = \prod_{i=1}^{d} \dfrac{A^* - \theta_i^* I}{\theta_0^* - \theta_i^*}$

(ii) $\quad A, A^*$ generate $\mathrm{Mat}_{d+1}(\mathbb{F})$

(iii) $\quad$ There does not exist $W \subseteq V$ s.t.
$W \ne 0$, $W \ne V$, $\quad AW \subseteq W$, $\quad A^* W \subseteq W$

pf $\quad$ Routine $\qquad \square$

Comments on TD pairs

Until further notice

V = vector space over $\mathbb{F}$ with finite positive dimension

Let A, A^* denote a TD pair on V

DEF 7 Let V' denote a vector space over $\mathbb{F}$ with finite pos dimension.
Let $A', A^{*\prime}$ denote a TD pair on V'.
By an <u>isomorphism of TD pairs</u> from A, A^* to $A', A^{*\prime}$ we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ that satisfies

$$A'\sigma = \sigma A, \qquad A^{*\prime}\sigma = \sigma A^*$$

— o —

An ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A is called <u>standard</u> whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

If $\{V_i\}_{i=0}^{d}$ is standard then so is $\{V_{d-i}\}_{i=0}^{d}$ and no further ordering is standard.

Similar concepts apply to A^*

DEF 8 A tridiagonal system (a TD-system) on V is a sequence

$$\Phi = \left(A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta} \right)$$

such that

(i) A, A^* is a TD pair on V

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A

(iii) $\{E_i^*\}_{i=0}^{\delta}$ is a standard ordering of the primitive idempotents of A^*

We say Φ is over $\mathbb{F}$. We call V the underlying vector space. We call A, A^* the associated TD pair

— o —

An isomorphism of TD systems is defined similar to Def 7

— o —

We mention some special cases of TD systems

Let Φ denote a TD system on V, as in DEF 8

One checks the following are equivalent:

(i) $d=0$
(ii) $\delta=0$
(iii) $A \in \mathbb{F} I$
(iv) $A^* \in \mathbb{F} I$
(v) $\dim V = 1$

Φ is called <u>trivial</u> whenever (i)-(v) hold

A <u>Leonard system</u> is a TD system such that

$$\dim E_i V = 1 \qquad 0 \le i \le d$$
$$\dim E_i^* V = 1 \qquad 0 \le i \le \delta$$

In this case $d = \delta = \dim V - 1$.

Given a TD system Φ on V, as in Def 8.

Then each of the following is a TD system on V:

$$\Phi^{\downarrow} := \left(A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_{\delta-i}\}_{i=0}^{\delta} \right),$$

$$\Phi^{\Downarrow} := \left(A; \{E_{d-i}\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{\delta} \right),$$

$$\Phi^{*} := \left(A^*; \{E^*_i\}_{i=0}^{\delta}; A; \{E_i\}_{i=0}^{d} \right)$$

Obs

$$\downarrow\Downarrow = \Downarrow\downarrow, \qquad \downarrow * = * \Downarrow \qquad \Downarrow * = * \downarrow$$

$$*^2 = 1, \qquad \downarrow^2 = 1, \qquad \Downarrow^2 = 1$$

The group gen by symbols $\downarrow, \Downarrow, *$ subject to the above relations is the Dihedral group D_4

D_4 has 8 elements. D_4 is the group of symmetries of a square.

So D_4 acts on the set of all TD systems

TD systems in the same D_4-orbit are called relatives.

Until further notice fix a TD system on V:

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$$

Obs

$$E_i^* A E_j^* = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i,j \le \delta)$$

$$E_i A^* E_j = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i,j \le d)$$

DEF 9 For $0 \le i \le d$ let θ_i denote the eigenvalue of A for E_i.

For $0 \le i \le \delta$ let θ_i^* denote the eigenvalue of A^* for E_i^*

Call $\{\theta_i\}_{i=0}^{d}$ the <u>eigenvalue</u> <u>sequence</u> of Φ

Call $\{\theta_i^*\}_{i=0}^{\delta}$ the <u>dual</u> <u>eigenvalue</u> <u>sequence</u> of Φ

obs

$$A = \sum_{i=0}^{d} \theta_i E_i, \qquad A^* = \sum_{i=0}^{\delta} \theta_i^* E_i^*$$

By construction

$\{\theta_i\}_{i=0}^{d}$ are mutually distinct scalars in $\mathbb{F}$

$\{\theta_i^*\}_{i=0}^{\delta}$ ...

DEF 10 Call Φ <u>bipartite</u> whenever

$$E_i^* A E_i^* = 0 \qquad (0 \le i \le \delta)$$

Call Φ <u>dual bipartite</u> whenever Φ^* is bipartite,

ie

$$E_i A^* E_i = 0 \qquad (0 \le i \le d)$$

Call Φ <u>totally bipartite</u> whenever Φ is bipartite and dual bipartite.

— o —

EXAMPLE 11 For Φ trivial,

Φ is bipartite iff $A = 0$,

Φ is dual bipartite iff $A^* = 0$

Φ is totally bip iff $A = A^* = 0$

Next goal: Classify the totally Bipartite TD systems up to isomorphism.

We use a conceptual approach with minimal computation.