LECTURE 14. FRIDAY OCT 4

We continue to discuss the DRG $\Gamma = (X, R)$

with diameter $D \geq 1$ and intersection numbers a_i, b_i, c_i.

Fix $x \in X$ and write $E_i^* = E_i^*(x)$, $A_i^* = A_i^*(x)$, $M^* = M^*(x)$ etc.

LEM 100 We have

(i) $A_0^* = I$

(ii) $\sum_{i=0}^{D} A_i^* = |X| E_0^*$

(iii) $\overline{A_i^*} = A_i^*$ $(0 \leq i \leq D)$

(iv) $A_i^{*t} = A_i^*$ $(0 \leq i \leq D)$

(v) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \leq i, j \leq D)$

pf (i) $A_0^* = |X| E_0^!$ $E_0 = |X|^{-1} J$

(ii) Apply ρ to

$$\sum_{i=0}^{D} E_i = I$$

(iii), (iv) clear

(v) Apply ρ to

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h$$

□

LEM 101 For $0 \le i,j \le D$

(i) $(E_i^*, E_j^*) = \delta_{ij} k_i$

(ii) $(A_i^*, A_j^*) = \delta_{ij} |X| k_i^*$

pf Use formula above Def 99 $\square$

LEM 102 For $0 \le h,i,j \le D$

$$q_{ij}^h = |X|^{-1} (k_h^*)^{-1} (A_i^* A_j^*, A_h^*)$$

pf Expand $A_i^* A_j^*$ using

$$A_i^* A_j^* = \sum_{l=0}^{D} q_{ij}^l A_l^*$$

and use LEM 101. $\square$

LEM 103 For $0 \le h, i, j \le D$

$$k_h^* q_{ij}^h = k_i^* q_{hj}^i = k_j^* q_{ih}^j$$

pf Obs

$$
\begin{aligned}
|X| k_h^* q_{ij}^h &= (A_i^* A_j^*, A_h^*) \\
&= (A_i^*, A_h^* \overline{A_j^*}^t) \\
&= (A_i^*, A_h^* A_j^*) \\
&= |X| k_i^* q_{hj}^i
\end{aligned}
$$

since these are real

Remaining assertions similarly obtained. □

thm 104 Fn $0 \le h, i, j \le D$ and

$0 \le r, s, t \le D$,

(i) $\left(E_i^* A_j E_h^*, \; E_r^* A_s E_t^* \right) = \delta_{ir} \delta_{js} \delta_{ht} k_h p_{ij}^h$

(ii) $\left(E_i A_j^* E_h, \; E_r A_s^* E_t \right) = \delta_{ir} \delta_{js} \delta_{ht} k_h^* q_{ij}^h$

pf (i) Routine counting

(ii) $LHS = tr\left(E_i A_j^* E_h \left(\overline{E_r A_s^* E_t} \right)^t \right)$

$= tr\left(E_i A_j^* E_h \; E_t A_s^* E_r \right)$ $\qquad tr(uv) = tr(vu)$

$= \delta_{ir} \delta_{ht} \; tr\left(E_i A_j^* E_h A_s^* \right)$

Also

$$tr\left(E_i A_j^* E_h A_s^* \right) = \sum_{y \in X} \sum_{z \in X} (E_i)_{yz} \underbrace{(A_j^*)_{zz}}_{= |X| (E_j)_{xz}} (E_h)_{zy} \underbrace{(A_s^*)_{yy}}_{= |X| (E_s)_{xy}}$$

$$= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_s)_{xy} \left(E_i \circ E_h \right)_{yz} \left(E_j \right)_{zx}$$

$$= |X|^2 \left((x,x)\text{-entry of } \; E_s \left(E_i \circ E_h \right) E_j \right)$$

$$= |X| \; trace\left(E_s \left(E_i \circ E_h \right) E_j \right)$$

$$= |X| \; trace\left(\left(E_i \circ E_h \right) E_j E_s \right)$$

$$= \delta_{js} |X| \; trace\left(\underbrace{\left(E_i \circ E_h \right)}_{= |X|^{-1} \sum_{l=0}^{D} q_{ih}^l E_l} E_j \right)$$

$$= \delta_{js}\, q_{ih}^{j}\, \mathrm{tr}(E_j)$$

$$= \delta_{js}\, q_{ih}^{j}\, k_j^*$$

$$= \delta_{js}\, k_h^*\, q_{ij}^{h}$$

$\square$

In thm 104 set

$r=i,\ s=j,\ t=h$ to get

$$\| E_i^* A_j E_h^* \|^2 = k_h\, p_{ij}^{h} \qquad 0 \le h,i,j \le D$$

$$\| E_i A_j^* E_h \|^2 = k_h^*\, q_{ij}^{h}$$

thm 105 (Krein condition)

$$q_{ij}^{h} \ge 0 \qquad 0 \le h,i,j \le D$$

pf Recall

$$\| B \|^2 \ge 0 \qquad \forall\, B \in \mathrm{Mat}_X(\mathbb{C})$$

Also $k_h^* = \mathrm{rank}\, E_h > 0$

$\square$

thm 106 (triple product relations)

Fix $0 \le h, i, j \le D$

(i) $E_i^* A_j E_h^* = 0$ iff $p^h_{ij} = 0$

(ii) $E_i A_j^* E_h = 0$ iff $q^h_{ij} = 0$

pf Recall $\|B\|^2 = 0$ iff $B = 0$ $\forall B \in Mat_X(\mathbb{C})$ $\square$

DEF 107 Γ is said to be <u>Q-polynomial</u>
(with respect to the given ordering of the primitive idempotents / eigenvalues of Γ)
whenever the following hold for $0 \le h, i, j \le D$:

(i) $q^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $q^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two

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Until further notice, assume Γ is Q-polynomial.

Write $A^* = A_1^*(x)$

We now show that A^* is a dual adjacency matrix with respect to x.

Define

$c_i^* = q^i_{i1}$ $\quad 1 \le i \le D,$ $\quad c_0^* = 0$

$a_i^* = q^i_{i1}$ $\quad 0 \le i \le D,$

$b_i^* = q^i_{i+1,1}$ $\quad 0 \le i \le D-1,$ $\quad b_D^* = 0$

Note

$c_i^* > 0$ $\quad 1 \le i \le D,$ $\quad c_1^* = 1,$

$b_i^* > 0$ $\quad 0 \le i \le D-1$ $\quad a_0^* = 0$

LEM 108 For $0 \le i \le D$,

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^*$$

where $A_{-1}^* = 0$, $A_{D+1}^* = 0$

pf This is just

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$$

with $j=1$. $\square$

Define polynomials $\{f_i^*\}_{i=0}^{D+1}$ in $\mathbb{C}[\lambda]$ by

$$f_0^* = 1, \qquad f_1^* = \lambda,$$

$$\lambda f_i^* = b_{i-1}^* f_{i-1}^* + a_i^* f_i^* + c_{i+1}^* f_{i+1}^* \qquad 1 \le i \le D-1$$

$$\lambda f_D^* = b_{D-1}^* f_{D-1}^* + a_D^* f_D^* + \frac{f_{D+1}^*}{c_1^* c_2^* \cdots c_D^*}$$

So for $0 \le i \le D$

f_i^* has degree i with

coef of λ^i in f_i^* is $\frac{1}{c_1^* c_2^* \cdots c_i^*}$

f_{D+1}^* is monic of degree $D+1$

By L108

$$f_i^*(A^*) = A_i^* \qquad 0 \le i \le D$$

$$f_{D+1}^*(A^*) = 0$$

so A^* generates M^*

Thm 109 Assume Γ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents of Γ. Fix $x \in X$ and write $A^* = A_1^*(x)$. Then A^* is a dual adjacency matrix of Γ wrt x and $\{E_i\}_{i=0}^{D}$.

Pf. Apply Def 4

We saw that A^* generates M^*.

Also for $0 \le i,j \le D$ such that $|i-j|>1$,

$$q_{ij}^{1} = 0$$

So

$$E_i A^* E_j = 0$$

by thm 106 $\square$