Next topic: the Krein parameters of a distance-regular graph.

Until further notice let $\Gamma = (X, R)$ denote a DRG with diameter $D \geq 1$ and intersection numbers a_i, b_i, c_i

Note that Γ is regular with valency $k = b_0$.
So k is an eigenvalue of Γ. Call this θ_0.
For $0 \leq i \leq D$ let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ have (y,z)-entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y,z) = i \\ 0 & \text{if } \partial(y,z) \neq i \end{cases} \qquad \forall y, z \in X$$

Call A_i the <u>ith</u> <u>distance</u> <u>matrix</u> of Γ.
For notational convenience define $A_{-1} = 0$, $A_{D+1} = 0$

Note that $\{A_i\}_{i=0}^D$ are lin indep. For $y \in X$ and $0 \leq i \leq D$

$$A_i \hat{y} = \sum_{z \in} \hat{z}$$

We have

$$A_0 = I$$

$$A_1 = A \quad (\text{adj matrix})$$

$$\sum_{i=0}^{D} A_i = J \quad (\text{all 1's matrix})$$

$$A_i^t = A_i \qquad 0 \le i \le D$$

LEM 91 For $0 \le i \le D$

$$A A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$$

pf Similar to pf of LEM 14

LEM 92

(i) $\{A_i\}_{i=0}^{D}$ is a basis for the Bose-Mesner algebra M

(ii) $\mathbb{C}$-vector space M has dimension $D+1$

(iii) Γ has exactly $D+1$ distinct eigenvalues

pf By L91

$\square$

Recall the polynomials $\{f_i\}_{i=0}^{D+1}$ from above LEM16.

Recall

$$f_0 = 1, \qquad f_1 = \lambda$$

$$\lambda f_i = b_{i-1} f_{i-1} + a_i f_i + c_{i+1} f_{i+1} \qquad 1 \le i \le D-1$$

$$\lambda f_D = b_{D-1} f_{D-1} + a_D f_D + \frac{f_{D+1}}{c_1 c_2 \cdots c_D}$$

By L91,

$$A_i = f_i(A) \qquad 0 \le i \le D$$

$$0 = f_{D+1}(A)$$

Since $\{A_i\}_{i=0}^{D}$ form a basis for M, $\exists$ scalars

$$p^h_{ij} \qquad 0 \le h, i, j \le D$$

such that

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \qquad 0 \le i, j \le D \qquad (*)$$

Since A_i, A_j commute,

$$p^h_{ij} = p^h_{ji} \qquad 0 \le h, i, j \le D$$

For $0 \le h \le D$ and $y, z \in X$ at $\partial(y, z) = h$, compute the (y, z)-entry in $(*)$ to find

$$p^h_{ij} = |\Gamma_i(y) \cap \Gamma_j(z)|$$

So p^h_{ij} is a nonneg integer

Note that

$c_i = p^i_{1 i-1}$ $\quad 1 \le i \le D$

$a_i = p^i_{1 i}$ $\quad 0 \le i \le D$

$b_i = p^i_{1 i+1}$ $\quad 0 \le i \le D-1$

We often call any p^h_{ij} an <u>intersection number</u> for Γ

LEM 93 For $0 \le h, i, j \le D$

$$k_h p^h_{ij} = k_i p^i_{hj} = k_j p^j_{ih}$$

pf Each of the 3 products equals $|X|^{-1}$ times the number of triples

$$\left| \{ xyz \mid x, y, z \in X, \ \partial(x,y) = h, \ \partial(y,z) = i, \ \partial(x,z) = j \} \right| \qquad \square$$

LEM 94 For $0 \le h, i, j \le D$

(i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other 2

(ii) $p^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other 2.

pf The distance function ∂ satisfies the triangle inequality. $\square$

— o —

For $B, C \in \mathrm{Mat}_X(\mathbb{C})$ define

$$(B, C) = \mathrm{trace}(B\overline{C}^t)$$

$$\|B\|^2 = (B, B)$$

Then $(\, ,\,)$ is a pos definite Hermitean form

This means

$$(\alpha B, C) = \alpha (B, C) \qquad \alpha \in \mathbb{C}$$

$$(B, C) = \overline{(C, B)}$$

$$(B + B', C) = (B, C) + (B', C)$$

$\|B\|^2$ is nonneg real number

$\|B\|^2 = 0$ iff $B = 0$

LEM 95 For $0 \le i, j \le D$

(i) $(A_i, A_j) = \delta_{ij} k_i |X|$

(ii) $(E_i, E_j) = \delta_{ij} k_i^*$

$\uparrow$ rank of E_i

pf (i)

$$\begin{aligned}
(A_i, A_j) &= (A_i, I A_j) \\
&= (A_i \bar{A}_j^t, I) \\
&= (A_i A_j, I) \\
&= \sum_{h=0}^{D} p_{ij}^h \underbrace{(A_h, I)}_{= \operatorname{tr} A_h = \delta_{h,0} |X|} \\
&= p_{ij}^0 |X| \\
&= \delta_{i,j} k_i |X|
\end{aligned}$$

(ii) $$(E_i, E_j) = (E_i, I E_j)$$
$$= (E_i \bar{E}_j^t, I)$$
$$= (E_i E_j, I)$$
$$= \delta_{ij} (E_i, I)$$
$$= \delta_{ij} \, \mathrm{tr}(E_i)$$
$$= \delta_{ij} k_i^*$$

□

LEM 96 For $0 \leq h, i, j \leq D$

$$p_{ij}^h = |X|^{-1} k_h^{-1} \langle A_i A_j, A_h \rangle$$

pf Expand $A_i A_j$ using
$$A_i A_j = \sum_{\ell=0}^{D} p_{ij}^{\ell} A_\ell$$

and use LEM 95 (i)

□

For $B, C \in \mathrm{Mat}_X(\mathbb{C})$ define

$$B \circ C \in \mathrm{Mat}_X(\mathbb{C})$$

by

$$(B \circ C)_{y,z} = B_{yz} C_{yz} \qquad y, z \in X$$

"entry-wise multiplication"

Obs

$$A_i \circ A_j = \delta_{ij} A_i \qquad 0 \le i, j \le D$$

So Bose-Mesner algebra M is closed under $\circ$

So $\exists$ scalars in $\mathbb{C}$:

$$q^h_{ij} \qquad 0 \le h, i, j \le D$$

such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad 0 \le i, j \le D$$

By constr

$$q^h_{ij} = q^h_{ji} \qquad 0 \le h, i, j \le D$$

The primitive idempotents have all entries in $\mathbb{R}$ so

$$q^h_{ij} \in \mathbb{R} \qquad 0 \le h, i, j \le D$$

We will show

$$q^h_{ij} \ge 0 \qquad 0 \le h, i, j \le D$$

The q^h_{ij} are the <u>Krein parameters</u> of Γ

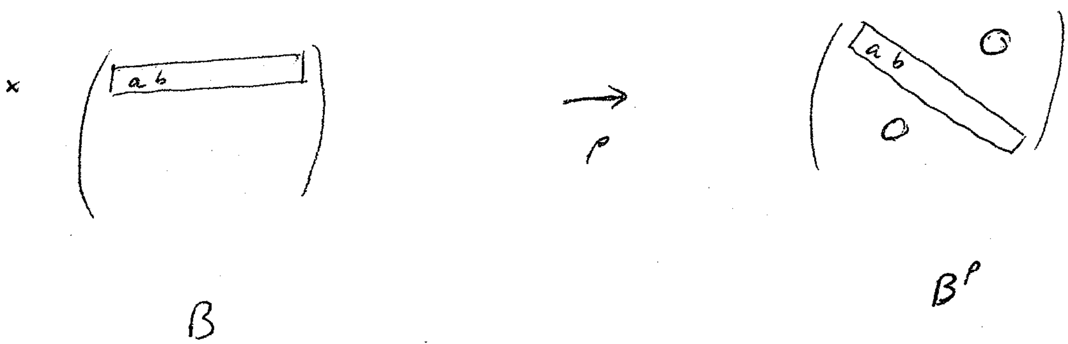
Until further notice fix $x \in X$

Recall $\quad E_i^* = E_i^*(x) \qquad 0 \le i \le D$

$\qquad\quad M^* = M^*(x), \qquad T = T(x)$

DEF 97 $\quad \forall\, B \in \mathrm{Mat}_X(\mathbb{C})$ let B^ρ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y,y)-entry

$$(B^\rho)_{yy} = B_{xy} \qquad \forall y \in X$$



LEM 98 $\quad$ We have

$$(B \circ C)^\rho = B^\rho C^\rho \qquad \forall\, B, C \in \mathrm{Mat}_X(\mathbb{C})$$

pf clear $\square$

obs

$$A_i^\rho = E_i^* \qquad 0 \le i \le 0$$

$$I^\rho = E_0^*$$

$$J^\rho = I$$

the restriction

$$\rho|_M : M \longrightarrow M^*$$

is an isomorphism of $\mathbb{C}$-vector spaces

[Caution: not iso of algebras]

One checks: $\forall\ B, C \in M$,

$$\langle B^\rho, C^\rho \rangle = |X|^{-1} \langle B, C \rangle$$

DEF 99 For $0 \le i \le 0$ define

$$A_i^* = |X| E_i^\rho$$

thus A_i^* is diagonal with (y,y)-entry

$$(A_i^*)_{yy} = |X| (E_i)_{xy}$$

call A_i^* the <u>ith</u> <u>dual</u> <u>distance-matrix</u> (with respect to given ordering $\{E_i\}_{i=0}^{0}$)

Note that $\{A_i^*\}_{i=0}^D$ is a basis for the $\mathbb{C}$-vector space M^*.