LECTURE 12 MONDAY SEPT. 30

We have considered the $\boxtimes$ module $V(t)$ of dim 2. We now consider more general $\boxtimes$-modules.

Fix an integer $D \geq 1$

Fix mutually distinct $t_1, t_2, \ldots, t_D \in \mathbb{C}$ s.t.

$$t_i \neq 0, \quad t_i \neq 1 \qquad 1 \leq i \leq D$$

Define $\boxtimes$-module

$$V = V(t_1) \otimes V(t_2) \otimes \cdots \otimes V(t_D)$$

So $\qquad \dim V = 2^D$

$\boxtimes$ acts on V as follows. For $x \in \boxtimes$

and $\quad u_1 \otimes u_2 \otimes \cdots \otimes u_D \in V$,

$x.(u_1 \otimes \cdots \otimes u_D) =$ $\qquad (*)$

$$\sum_{i=1}^{D} u_1 \otimes \cdots \otimes u_{i-1} \otimes x u_i \otimes u_{i+1} \otimes \cdots \otimes u_D$$

LEM 83 Each $\boxtimes$-generator x_{rs} is diagonalizable on V, with eigenvalues

$$D-2i \qquad 0 \le i \le D$$

For $0 \le i \le D$ the eigenspace for $D-2i$ has dimension

$$\binom{D}{i}$$

Pf Recall x_{rs} acts on $V(t)$ in a diagonalizable fashion, with eigenvalues $1, -1$.

Now use $*$. $\square$

Recall the group $G \subseteq S_4$ from LEM 68

Recall the related elements $g, g', g'' \in \mathrm{End}(\underline{V}(t))$

from LEM 67.

Define $g, g', g'' \in \mathrm{End}(V)$ s.t.

$$g.(u_1 \otimes u_2 \otimes \cdots \otimes u_D) = (g.u_1) \otimes (g.u_2) \otimes \cdots \otimes (g.u_D)$$

$$\forall\ u_i \in \underline{V}(t_i) \qquad 1 \le i \le D$$

and similar for g', g''.

Observe LEM 68 still holds. Thus for distinct

$i, j \in \underline{\mathbb{I}}$ and $\sigma = (01)(23) \in G$

$$X_{\sigma(i), \sigma(j)} = g\, X_{ij}\, g^{-1} \quad \text{on } V$$

and similar for g', g''.

Consider

$$A^* = x_{10} \qquad A = x_{23}$$

One checks A, A^* act on V as a TD pair

Next goal: show how to recover $t_1, \ldots, t_D$

from A, A^*.

For $0 \le i \le D$ let E_i (resp E_i^*) denote the

prim idempotent of A (resp. A^*) for the eigval

$D - 2i$

LEM 84

$$\operatorname{trace}(E_0 E_0^*) = \prod_{i=1}^{D} (1-t_i)$$

pf Recall the subspaces a, b, c, d for $V(t)$ from Lec 10

Recall vectors

$$\begin{array}{cccc} a & b & c & d \\ \hline u & v & w & r \end{array}$$

V has basis

$v_1 \otimes v_2 \otimes \cdots \otimes v_D \qquad v_i \in \{u, v\} \qquad 1 \le i \le D \qquad ☆$

Consider the matrices that rep E_0, E_0^* rel ☆

$$E_0^* : \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)$$

where the "1" is in row/col that corresponds to $u \otimes u \otimes \cdots \otimes u$

So

$$\text{trace}(E_0 E_0^*) = (x, x)\text{-entry of } E_0$$

where

$$x = u \otimes u \otimes \cdots \otimes u$$

Find the entry. Show $E_0^* E_0 x = x \prod_{i=1}^{D} (1 - \theta_i)$.

Recall $A^* = X_{10}$ $A = X_{23}$

Consider $\sigma = (03)(12) \in G$

So

$$A = X_{23} = X_{\sigma(1)\sigma(0)}$$
$$= g'' X_{10} (g'')^{-1}$$
$$= g'' A^* (g'')^{-1}$$

Also

$$E_0 = \prod_{i=1}^{D} \frac{A - \theta_i}{\theta_0 - \theta_i} \qquad \theta_i = D - 2i$$

$$E_0^* = \prod_{i=1}^{D} \frac{A^* - \theta_i^*}{\theta_0^* - \theta_i^*} \qquad \theta_i^* = D - 2i$$

So

$$E_0 = g'' E_0^* (g'')^{-1}$$

Recall by LEM 67, on $V(t)$,

$$g'' u = r, \qquad (g'')^2 = (1-t) I$$

$$(g'')^{-1} = \frac{1}{1-t} g''$$

So

$$E_0^* E_0 . \ u \otimes u \otimes \cdots \otimes u$$

$$= E_0^* g'' E_0^* (g'')^{-1} \ u \otimes u \otimes \cdots \otimes u$$

$$= E_0^* g'' E_0^* \Big((g'')^{-1} u \otimes (g'')^{-1} u \otimes \cdots \otimes (g'')^{-1} u \Big)$$

$$= E_0^* g'' E_0^* \ (r \otimes r \otimes \cdots \otimes r) \prod_{i=1}^{D} \frac{1}{1-t_i}$$

$$\left[\begin{array}{l} \text{on } V(t) \qquad r = (1-t)u - tv \\ \text{so} \\ E_0^*(r \otimes r \otimes \cdots \otimes r) = u \otimes u \otimes \cdots \otimes u \prod_{i=1}^{D} (1-t_i) \end{array} \right]$$

$$= E_0^* g'' \ (u \otimes u \otimes \cdots \otimes u)$$

$$= E_0^* (g'' u \otimes g'' u \otimes \cdots \otimes g'' u)$$

$$= E_0^* (r \otimes r \otimes \cdots \otimes r)$$

$$= u \otimes \cdots \otimes u \prod_{i=1}^{D} (1-t_i) \qquad \checkmark \qquad \square$$

Pick nonzero $z \in \mathbb{C}$ s.t.

$$t_i z \neq 1 \qquad 1 \leq i \leq D$$

Consider $\boxtimes$-module

$$V^{(z)} = \mathbb{V}(t_1 z) \otimes \mathbb{V}(t_2 z) \otimes \cdots \otimes \mathbb{V}(t_D z)$$

Write

$$A^{*(z)} = \text{action of } x_{10} \text{ on } V^{(z)}$$

$$A^{(z)} = \quad \text{''} \quad x_{23} \quad \text{''}$$

Similarly def

$$E_i^{(z)}, \quad E_i^{*(z)} \qquad 0 \leq i \leq D.$$

Define

$$P(z) = \text{trace}\left(E_0^{(z)} E_0^{*(z)}\right)$$

By LEM 84

$$P(z) = \prod_{i=1}^{D} (1 - t_i z)$$

Call $P(z)$ the <u>Drinfeld poly</u> of $V = V^{(1)}$

We now compute $P(z)$ in terms of the original $\boxtimes$-module V.

Using the notation of Prop 64, identify

$$u = \begin{pmatrix}1\\0\end{pmatrix} \qquad v = \begin{pmatrix}-1\\-1\end{pmatrix} \qquad w = \begin{pmatrix}0\\1\end{pmatrix} \qquad r = \begin{pmatrix}1\\zt\end{pmatrix}$$

As $\mathbb{C}$-vector spaces

$$V^{(z)} = V = \underbrace{(\mathbb{C}^2) \otimes (\mathbb{C}^2) \otimes \cdots \otimes (\mathbb{C}^2)}_{D}$$

u, v, w are indep of z but r is not

On V

X_{01}, X_{12}, X_{20} are indep of z

X_{30}, X_{31}, X_{32} depend on z

Observe

$$A^{*(z)} = \text{action of } X_{10} \text{ on } V$$

[indep of z]

$$= A^*$$

LEM 85

$$A^{(z)} = zA + (1-z)\,X_{20}$$

pf First check for $p=1$.

In the notation of Prop 64,

$$A = X_{23} : \begin{pmatrix} 1 & 0 \\ 2t & -1 \end{pmatrix}$$

$$A^{(z)} : \begin{pmatrix} 1 & 0 \\ 2zt & -1 \end{pmatrix}$$

$$X_{20} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It works ✓

Now for general p,

For $u_1 \otimes u_2 \otimes \cdots \otimes u_p \in V$,

$$A^{(z)} . (u_1 \otimes u_2 \otimes \cdots \otimes u_p) = \sum_{i=1}^{p} u_1 \otimes \cdots \otimes u_{i-1} \otimes \underbrace{A^{(z)} u_i}_{\substack{\| \\ (zA + (1-z)X_{20}) u_i}} \otimes u_{i+1} \otimes \cdots \otimes u_p$$

$$= (zA + (1-z)\,X_{20}) . (u_1 \otimes u_2 \otimes \cdots \otimes u_p)$$

$\square$

Helpful to view

$$A^{*(z)} = A^* = (X_{10} + X_{02}) + X_{20}$$

$$A^{(z)} = zA + (1-z)X_{20}$$

$$= z(A + X_{02}) + X_{20}$$

$$= z(X_{02} + X_{23}) + X_{20}$$

We now consider the actions of

$$X_{10} + X_{02}, \qquad X_{02} + X_{23}$$

on the eigenspaces of $\quad X_{20}$

For $0 \le i \le D$ define

U_i = eigenspace of X_{20} for the eigenvalue $D-2i$

Define $U_{-1} = 0$, $U_{D+1} = 0$

Just as in thm 53,

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_D V)$$

$$0 \le i \le D$$

So $U_0 = E_0^* V$

LEM 86 For $0 \le i \le D$,

(i) $(X_{02} + X_{23}) U_i \subseteq U_{i+1}$

(ii) $(X_{10} + X_{02}) U_i \subseteq U_{i+1}$

pf (i) One checks

$$[X_{20}, X_{02} + X_{23}] = -2(X_{02} + X_{23})$$

(ii) One checks

$$[X_{20}, X_{10} + X_{02}] = 2(X_{10} + X_{02})$$

$\square$

Define

$$e^{+} = \frac{X_{02} + X_{23}}{2}$$

$$e^{-} = \frac{X_{20} + X_{01}}{2}$$

For $0 \le i \le D$ consider the composition

$$U_0 \xrightarrow{(e^{+})^{i}} U_i \xrightarrow{(e^{-})^{i}} U_0$$

Recall $U_0 = E_0^{*}V$ has dim 1.

Define

$\varphi_i =$ eigenvalue of $(e^{-})^{i}(e^{+})^{i}$ on U_0

Note $\varphi_0 = 1$. Call the sequence $\{\varphi_i\}_{i=0}^{d}$ the

<u>split sequence</u> for the $\boxtimes$-module V.

LEM 87 For $0 \le i \le j \le D$,

(i) On U_i,

$$\left(A^{(z)} - \theta_{j-1} I\right) \cdots \left(A^{(z)} - \theta_{i+1} I\right)\left(A^{(z)} - \theta_i I\right) = \left(2z e^+\right)^{j-i}$$

$\theta_i = D - 2i$

(ii) On U_j,

$$\left(A^* - \theta^*_{i+1} I\right) \cdots \left(A^* - \theta^*_{j-1} I\right)\left(A^* - \theta^*_j I\right) = (-1)^{j-i}\left(2e^-\right)^{j-i}$$

$\theta^*_i = D - 2i$

pf (i) On U_i

$$A^{(z)} = z\underbrace{\left(x_{02} + x_{23}\right)}_{2e^+} + \underbrace{x_{20}}_{\theta_i}$$

so

$$A^{(z)} - \theta_i I = 2z e^+$$

Also by LEM 86 $e^+ U_i \subseteq U_{i+1}$

(ii) On U_i

$$A^* = \underbrace{\left(x_{10} + x_{02}\right)}_{= -2e^-} + \underbrace{x_{20}}_{= \theta^*_i}$$

so

$$A^* - \theta^*_i = -2e^-$$

Also by LEM 86 $e^- U_i \subseteq U_{i-1}$ $\square$

LEM 88. For $0 \le i \le D$,

$$E_0^* = (-1)^i \frac{(e^-)^i}{i!} \quad \text{on } U_i$$

pf Recall

$$E_0^* = \prod_{j=1}^{D} \frac{A^* - \theta_j^*}{\theta_0^* - \theta_j^*} \qquad \theta_j^* = D - 2j$$

$$= \prod_{j=1}^{D} \frac{1}{\theta_0^* - \theta_j^*} \prod_{j=i+1}^{D} (A^* - \theta_j^*) \prod_{j=1}^{i} (A^* - \theta_j^*)$$

For $v \in U_i$

$$E_0^* v = \prod_{j=1}^{D} \frac{1}{\theta_0^* - \theta_j^*} \prod_{j=i+1}^{D} (A^* - \theta_j^*) \underbrace{\prod_{j=1}^{i} (A^* - \theta_j^*) v}_{(-1)^i (2e^-)^i v \ \in U_0}$$

[on $U_0 = E_0^* V$ A^* acts as $\theta_0^* I$]

$$= \underbrace{\prod_{j=1}^{D} \frac{1}{\theta_0^* - \theta_j^*} \prod_{j=i+1}^{D} (\theta_0^* - \theta_j^*)}_{= \prod_{j=1}^{i} \frac{1}{\theta_0^* - \theta_j^*} = \frac{1}{2^i i!}} (-1)^i (2e^-)^i v$$

$\square$

Aside

LEM 89 Let $\theta_0, \theta_1, \ldots, \theta_D$ denote any scalars in $\mathbb{C}$ and define polynomials

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}) \qquad 0 \le i \le D$$

$$\eta_i = (\lambda - \theta_D)(\lambda - \theta_{D-1}) \cdots (\lambda - \theta_{D-i+1})$$

Then

$$\eta_D = \sum_{i=0}^{D} \eta_{D-i}(\theta_0)\, \tau_i$$

pf Use ind on D $\square$

thm 90 The Drinfeld polynomial satisfies

$$P(z) = \sum_{i=0}^{D} \frac{\varphi_i (-1)^i z^i}{(i!)^2}$$

where $\{\varphi_i\}_{i=0}^{D}$ is the split sequence of $\boxtimes$-module V.

pf By def.

$$P(z) = \text{trace}\left(E_0^* E_0^{(z)} \right)$$

$$= \text{eigval of } E_0^* E_0^{(z)} E_0^* \quad \text{on} \quad E_0^* V = U_0$$

Obs

$$E_0^{(z)} = \prod_{i=1}^{D} \frac{A^{(z)} - \theta_i I}{\theta_0 - \theta_i} \qquad \theta_i = D - 2i$$

$$= \frac{\eta_0 (A^{(z)})}{\eta_0 (\theta_0)}$$

$$\overset{L\,89}{=} \sum_{i=0}^{D} \frac{\eta_{D-i} (\theta_0) \, \tau_i (A^{(z)})}{\eta_0 (\theta_0)}$$

For $v \in U_0$

$$E_0^* E_0^{(z)} E_0^* v = E_0^* E_0^{(z)} v$$

$$= \sum_{i=0}^{D} \underbrace{\frac{\eta_{D-i}(\theta_0)}{\eta_0(\theta_0)}}_{\substack{= \\ \frac{1}{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_i)} \\ = \\ \frac{1}{2^i\, i!}}} \underbrace{E_0^* \underbrace{T_i(A^{(z)})\, v}_{\substack{= \ L87(i) \\ (2ze^+)^i v \ \in U_i}}}_{\substack{= \ L88 \\ \underbrace{\frac{(-1)^i (e^-)^i}{i!} (2ze^+)^i v}_{\substack{= \text{ def of } \varphi_i \\ \frac{\varphi_i (-1)^i 2^i z^i}{i!}}} \ \in U_0}}$$

$$= \sum_{i=0}^{D} \frac{\varphi_i (-1)^i z^i}{(i!)^2}$$

$\square$