# LECTURE 11 FRIDAY SEPT. 27

Recall our $\boxtimes$-module $V = V(\epsilon)$ of dim 2

We continue to discuss the operators $H, H', H''$

in End$(V)$. We saw

$$[H, H'] = 2i H'' \qquad [H', H''] = 2i H$$

$$[H'', H] = 2i H'$$

The $H, H', H''$ are not uniquely defined. We may change the sign of any two.

On $V$ each of $H, H', H''$ has eigenvalues $1, -1$ and trace $0$

So view $H, H', H'' \in sl_2$. They are lin indep.

We call $H, H', H''$ the <u>Dirac</u> <u>basis</u> for $sl_2$.

With respect to an appropriate basis for $V$,

$$H: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H': \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H'': \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Recall the invertible elements $g, g', g'' \in \mathrm{End}(V)$

recall
$H \in \mathbb{C}g, \quad H' \in \mathbb{C}g', \quad H'' \in \mathbb{C}g''$

LEM 7.5 We have

| $u$ | $H$ | $H'$ | $H''$ |
|---|---|---|---|
| $gug^{-1}$ | $H$ | $-H'$ | $-H''$ |
| $g'u(g')^{-1}$ | $-H$ | $H'$ | $-H''$ |
| $g''u(g'')^{-1}$ | $-H$ | $-H'$ | $H''$ |

pf Use L7.1 $\square$

Next goal

Find $\alpha, \beta, \gamma \in \mathbb{C}$ s.t.

$$H = \alpha g \qquad H' = \beta g' \qquad H'' = \gamma g'' \qquad *$$

LEM 76 $\alpha, \beta, \gamma$ satisfy

(i) $\alpha^2 = \dfrac{1}{1-t^{-1}} \qquad \beta^2 = \dfrac{1}{t} \qquad \gamma^2 = \dfrac{1}{1-t}$

(ii) $\alpha\beta\gamma = \dfrac{i}{t-1}$

pf (i) Take det in $*$

Each of $H, H', H''$ has eigvals $1, -1$ and det $= -1$

By LEM 67

$\det g = t^{-1} - 1$,

$\det g' = -t$,

$\det g'' = t - 1$

(ii) By $*$

$$\underbrace{HH'H''}_{\overset{\shortparallel}{iI}} = \alpha\beta\gamma \underbrace{gg'g''}_{\overset{\shortparallel}{(t-1)I}} \qquad \text{LEM 72}$$

$\square$

To solve for $\alpha, \beta, \gamma$ in terms of $t$

need the square root of $t$ and $1-t$

Change vars (motivation)

Write
$$t = u^2$$
$$1-t = v^2$$

So
$$1 = u^2 + v^2$$
$$= \underbrace{(u+iv)}_{= s}\underbrace{(u-iv)}_{= s^{-1}}$$

$$u + iv = s$$
$$u - iv = s^{-1}$$

$$u = \frac{s+s^{-1}}{2} \qquad v = \frac{s-s^{-1}}{2i}$$

So
$$t = u^2 = \frac{(s+s^{-1})^2}{4}$$
$$1-t = v^2 = -\frac{(s-s^{-1})^2}{4}$$

DEF 77 For our $\boxtimes$-module $\mathbb{V}(t)$

define $s \in \mathbb{C}$ to be any solution to

$$t = \frac{(s+s^{-1})^2}{4}$$

[so $s$ is defined up to sign and reciprocal]

We obs

$s \neq 0, \quad s \neq \pm 1, \quad s \neq \mp i$

LEM 78 $H, H', H''$ can be chosen such that

$$\alpha = \frac{s + s^{-1}}{s - s^{-1}}$$

$$\beta = \frac{2}{s + s^{-1}}$$

$$\gamma = \frac{2i}{s - s^{-1}}$$

pf The above values meet the requirements of LEM 76. $\square$

— o —

We now write each $\boxtimes$-gen $x_{ij}$ in terms of $H, H', H''$

Prop 79 We write each $\boxtimes$-generator $x_{ij}$ in terms of $H, H', H''$.

| | $H$ | $H'$ | $H''$ |
|---|---|---|---|
| $x_{01}$ | $0$ | $\frac{s+s^{-1}}{2}$ | $i\,\frac{s-s^{-1}}{2}$ |
| $x_{23}$ | $0$ | $\frac{s+s^{-1}}{2}$ | $-i\,\frac{s-s^{-1}}{2}$ |
| $x_{02}$ | $-\frac{s+s^{-1}}{s-s^{-1}}$ | $0$ | $-i\,\frac{2}{s-s^{-1}}$ |
| $x_{13}$ | $-\frac{s+s^{-1}}{s-s^{-1}}$ | $0$ | $i\,\frac{2}{s-s^{-1}}$ |
| $x_{03}$ | $-\frac{s-s^{-1}}{s+s^{-1}}$ | $\frac{2}{s+s^{-1}}$ | $0$ |
| $x_{12}$ | $-\frac{s-s^{-1}}{s+s^{-1}}$ | $-\frac{2}{s+s^{-1}}$ | $0$ |

pf For each row of the above table, write each term in the notation of Prop 64 using LEM 67 and LEM 78. $\square$

COR 80



| | $H$ | $H'$ | $H''$ |
|---|---|---|---|
| $\frac{X_{01}+X_{23}}{2}$ | $0$ | $\frac{s+s^{-1}}{2}$ | $0$ |
| $\frac{X_{01}-X_{23}}{2}$ | $0$ | $0$ | $i\,\frac{s-s^{-1}}{2}$ |
| $\frac{X_{02}+X_{13}}{2}$ | $-\frac{s+s^{-1}}{s-s^{-1}}$ | $0$ | $0$ |
| $\frac{X_{02}-X_{13}}{2}$ | $0$ | $0$ | $-i\,\frac{2}{s-s^{-1}}$ |
| $\frac{X_{03}+X_{12}}{2}$ | $-\frac{s-s^{-1}}{s+s^{-1}}$ | $0$ | $0$ |
| $\frac{X_{03}-X_{12}}{2}$ | $0$ | $\frac{2}{s+s^{-1}}$ | $0$ |

pf use Prop 79 $\square$

## An aside

Consider the Dirac basis $H, H', H''$ for $sl_2$

View $s$ as an indet

obs

$$\frac{2}{s - s^{-1}} = \frac{1}{s+1} + \frac{1}{s-1}$$

$$\frac{2}{s+s^{-1}} = \frac{1}{s+i} + \frac{1}{s-i}$$

By Prop 79 we view each $\boxtimes$-generator $x_{ij}$ as an element of Lie algebra

$$sl_2 \otimes \mathbb{C}\left[s, s^{-1}, \frac{1}{s+1}, \frac{1}{s-1}, \frac{1}{s+i}, \frac{1}{s-i}\right] \quad (\star)$$

Open Problem Find an attractive presentation of the Lie algebra $(\star)$ by generators + relations.

Return to Dirac basis $H, H', H''$ for $sl_2$.

Recall the group $G \subseteq S_4$.

$G$ acts on $sl_2$ as a group of auts

| | image under $\sigma$ | | |
|---|---|---|---|
| $\sigma$ | $H$ | $H'$ | $H''$ |
| $(01)(23)$ | $H$ | $-H'$ | $-H''$ |
| $(02)(13)$ | $-H$ | $H'$ | $-H''$ |
| $(03)(12)$ | $-H$ | $-H'$ | $H''$ |

Next goal: express each element of $G$ in terms of exponentials.

View

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$$

$$H' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e + f$$

$$H'' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = if - ie$$

So

$$e = \frac{H' + iH''}{2} \qquad f = \frac{H' - iH''}{2}$$

Define

$$e' = \frac{H'' + iH}{2} \qquad f' = \frac{H'' - iH}{2}$$

$$e'' = \frac{H + iH'}{2} \qquad f'' = \frac{H - iH'}{2}$$

Recall $\forall\ 0 \neq z \in \mathbb{C}$

$$\exp(ze)\ \exp\left(-\tfrac{1}{z}f\right)\ \exp(ze)$$

$$= \begin{pmatrix} 0 & z \\ -1/z & 0 \end{pmatrix}$$

$$= \exp\left(-\tfrac{1}{z}f\right)\ \exp(ze)\ \exp\left(\tfrac{-1}{z}f\right)$$

Call this $E_z$

Similary define $E_z'$, $E_z''$ using

$e', f'$ and $e'', f''$

| $z$ | $1$ | $-1$ | $i$ | $-i$ |
|---|---|---|---|---|
| $E_z$ | $iH''$ | $-iH''$ | $iH'$ | $-iH'$ |
| $E_z'$ | $iH$ | $-iH$ | $iH''$ | $-iH''$ |
| $E_z''$ | $iH'$ | $-iH'$ | $iH$ | $-iH$ |

pf  Fa $z=1$

$$E_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iH''$$

other entries sim. □

thm 82 the automorphism $(01)(23)$ of $sl_2$ is equal to each of

$$\exp \operatorname{ad} \frac{iH+H'}{2} \quad \exp \operatorname{ad} \frac{iH-H'}{2} \quad \exp \operatorname{ad} \frac{iH+H'}{2}$$

$$\exp \operatorname{ad} \frac{iH-H'}{2} \quad \exp \operatorname{ad} \frac{iH+H'}{2} \quad \exp \operatorname{ad} \frac{iH-H'}{2}$$

$$\exp \operatorname{ad} \frac{iH+H''}{2} \quad \exp \operatorname{ad} \frac{iH-H''}{2} \quad \exp \operatorname{ad} \frac{iH+H''}{2}$$

$$\exp \operatorname{ad} \frac{iH-H''}{2} \quad \exp \operatorname{ad} \frac{iH+H''}{2} \quad \exp \operatorname{ad} \frac{iH-H''}{2}$$

To get the auts $(02)(13)$ and $(03)(12)$, cyclically permute $H \to H' \to H'' \to H$ in the above formulas.

pf $\forall u \in sl_2$

$$\exp \operatorname{ad} \frac{iH+H'}{2} \exp \operatorname{ad} \frac{iH-H'}{2} \exp \operatorname{ad} \frac{iH+H'}{2} (u)$$

$$\overset{\text{on } \mathbb{C}^2}{=} \underbrace{\Big(\exp \underbrace{\frac{iH+H'}{2}}_{\frac{-1}{i}f''} \exp \underbrace{\frac{iH-H'}{2}}_{ie''} \exp \frac{iH+H'}{2}\Big)}_{E_i''} u \Big( \qquad \Big)^{-}$$

$$= E_i'' \, u \, (E_i'')^{-1}$$

$$= iH \, u \, (iH)^{-1}$$

$$= H u H^{-1}$$

the map $u \to HuH^{-1}$ sends

$$H \to H, \qquad H' \to -H' \qquad H'' \to -H''$$

so this agrees with $(01)(23)$.

The other entries are similar. $\square$