LECTURE 10 WED SEPT. 25

Recall our $\boxtimes$-module V:

V is a vector space over $\mathbb{C}$ with dim 2

Pick any 4 mutually distinct 1 dim'l subspaces a, b, c, d of V

$$\mathbb{I} = \{ \underset{0}{a}, \underset{1}{b}, \underset{2}{c}, \underset{3}{d} \}$$

For distinct $i, j \in \mathbb{I}$ $\quad x_{ij} \in \boxtimes$ acts on V s.t.

subspace i is eigenspace for x_{ij} with eigenval -1

... j ... 1

One checks the $\boxtimes$-module V is irreducible.

Pick a non0 vector in each of a, b, c, d:

space	a	b	c	d
vector	u	v	w	r

WLOG $\quad u + v + w = 0$

u, w is basis for V

Relative this basis

$u \quad v \quad w \quad r$

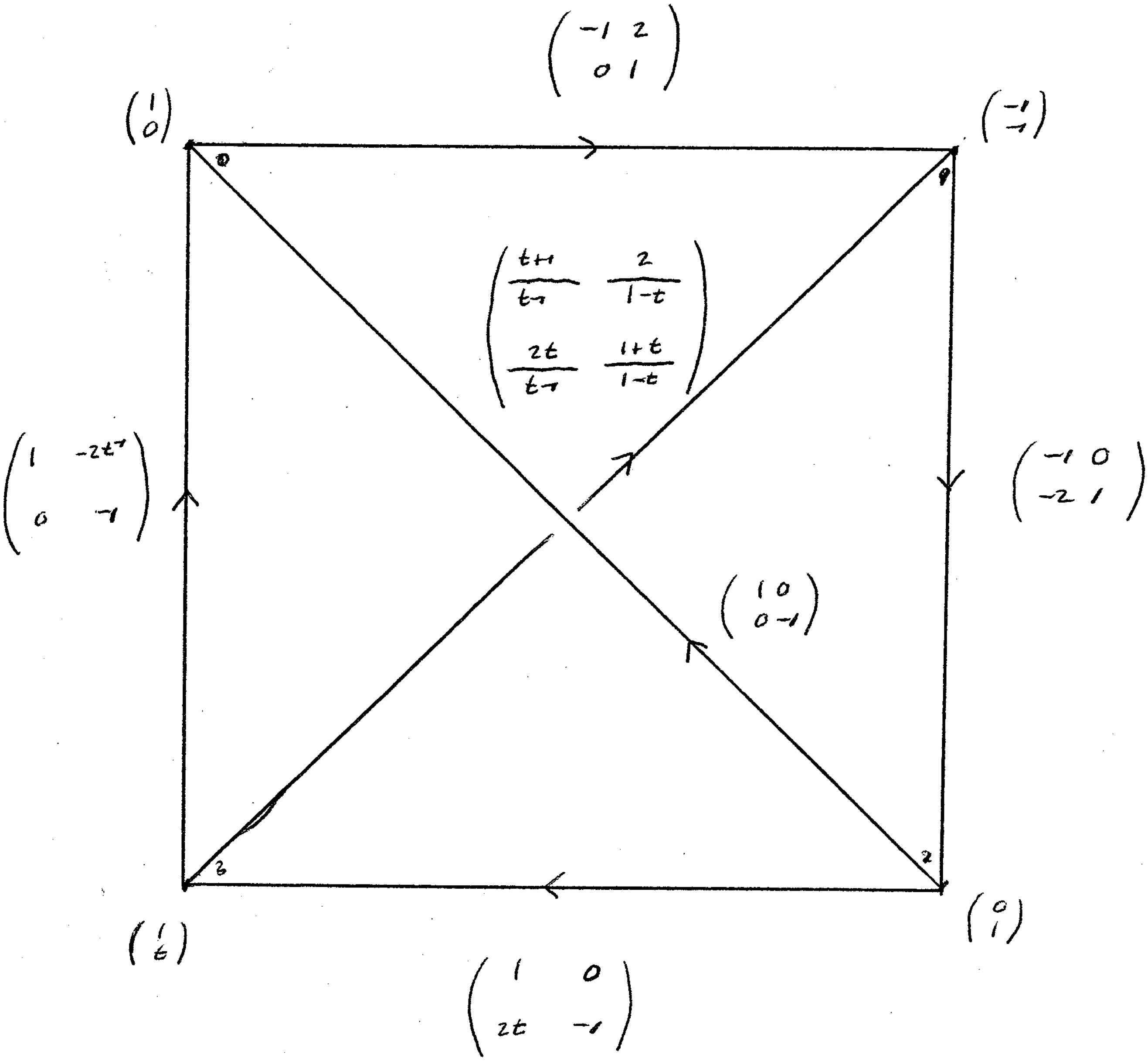
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ t \end{pmatrix}$

where
$t \in \mathbb{C} \quad t \neq 0, \ t \neq 1$

Call t the <u>evaluation parameter</u> of the $\boxtimes$-module V.

We now give the matrices that represent the $\boxtimes$ generators x_{ij} with respect to the basis u, w

Prop 64 with above notation



pf the given matrices have the required eigenvalues and eigenvectors. $\square$

By Prop 64 the $\boxtimes$-module V is determined up to isomorphism by t.

the $\boxtimes$-module will be denoted $\mathbb{V}(t)$.

$\mathbb{V}(t)$ exists $\forall t \in \mathbb{C}$ s.t. $t \neq 0$, $t \neq 1$.

LEM 65 On the $\boxtimes$-module $\mathbb{V}(t)$, the following are 0:

$$x_{01} + (t-1)\, x_{02} - t\, x_{03}$$

$$x_{10} + (t-1)\, x_{13} - t\, x_{12}$$

$$x_{23} + (t-1)\, x_{20} - t\, x_{21}$$

$$x_{32} + (t-1)\, x_{31} - t\, x_{30}$$

pf use Prop 64. $\square$

In Prop 64. it is useful to replace t by an indeterminate T.

let $\mathbb{C}[T, T^{-1}, (T-1)^{-1}]$ denote the $\mathbb{C}$-algebra of all Laurent polys in T, $T-1$ that have all coeffs in $\mathbb{C}$.

Abbr $\mathcal{A} = \mathbb{C}[T, T^{-1}, (T-1)^{-1}]$

the $\mathbb{C}$-vector space $\mathfrak{sl}_2 \otimes \mathcal{A}$ consists of the 2×2 matrices with entries in $\mathcal{A}$ and trace 0.

$\mathfrak{sl}_2 \otimes \mathcal{A}$ becomes a Lie algebra with

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab \qquad u, v \in \mathfrak{sl}_2 \qquad a, b \in \mathcal{A}$$

"3 point $\mathfrak{sl}_2$-loop algebra"

In Prop 64, the replacement $t \to T$

induces map $x_{ij} \to \mathfrak{sl}_2 \otimes \mathcal{A}$

which induces a Lie algebra hom

$$\boxtimes \longrightarrow \mathfrak{sl}_2 \otimes \mathcal{A} \qquad (*)$$

It turns out $(*)$ is an isomorphism.

So the Lie algebra $\boxtimes$ is iso to the

3-point $\mathfrak{sl}_2$-loop algebra. See

Hartwig and Terwilliger

the tetrahedron algebra, the Onsager algebra, and

the $\mathfrak{sl}_2$ loop algebra.

Back in LEM 65, note that the real parameter t is invariant under these moves:

$$\begin{array}{ccc} 0 \leftrightarrow 1 & \qquad & \\ 3 \leftrightarrow 2 & & \end{array} \qquad \begin{tikzcd} 0 \arrow[dr, leftrightarrow] & 1 \arrow[dl, leftrightarrow] \\ 3 & 2 \end{tikzcd} \qquad \begin{array}{cc} 0 & 1 \\ \updownarrow & \updownarrow \\ 3 & 2 \end{array}$$

We now explain this.

Recall the subspaces a, b, c, d of $\mathbb{V}(t) = V$.

Prop 66 $\exists$ invertible

g in $\mathrm{End}(V)$ that sends $a \leftrightarrow b$ and $c \leftrightarrow d$

g' ... $a \leftrightarrow c$ ... $b \leftrightarrow d$

g'' ... $a \leftrightarrow d$ .. $b \leftrightarrow c$

pf Consider g'.

Adopt the notation of Prop 64.

Define $g' = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$ so $g'^2 = tI$

subspace	a	b	c	d
basis	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ t \end{pmatrix}$
image under g'	$\begin{pmatrix} 0 \\ t \end{pmatrix}$	$\begin{pmatrix} -1 \\ -t \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} t \\ t \end{pmatrix}$
	$\uparrow$ c	$\uparrow$ d	$\uparrow$ a	$\uparrow$ b

So g' exists. By symmetry g, g'' exist.

$\square$

Note each of g, g', g'' is defined up to a nonzero scalar factor.

LEM 67 In the notation of Prop 64,

$$g: \begin{pmatrix} 1 & -t^{-1} \\ 1 & -1 \end{pmatrix}$$

$$g': \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

$$g'': \begin{pmatrix} 1 & -1 \\ t & -1 \end{pmatrix}$$

pf One checks that the given matrices satisfy the conditions in Prop 66. $\square$

Recall the symmetric group S_4.

View S_4 as the group of perms of $\mathbb{I} = \{0,1,2,3\}$

Let G denote the subgroup of S_4 consisting of

$(01)(23), \quad (02)(13), \quad (03)(12)$

and the identity.

G is normal in S_4.

G is iso to Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

LEM 68 Pick distinct $i, j \in \mathbb{I}$.

(i) For $\sigma = (01)(23) \in G$,

$$x_{\sigma(i)\sigma(j)} = g\, x_{ij}\, g^{-1} \qquad \text{on } V(t)$$

(ii) For $\sigma = (02)(13) \in G$,

$$x_{\sigma(i)\sigma(j)} = g'\, x_{ij}\, (g')^{-1} \qquad \text{on } V(t)$$

(iii) For $\sigma = (03)(12) \in G$,

$$x_{\sigma(i)\sigma(j)} = g''\, x_{ij}\, (g'')^{-1} \qquad \text{on } V(t)$$

pf (i) In the equation

each side acts on $g(1)$ and $g(2)$

as -1 and 1, respectively.

(ii), (iii) sim. $\square$

It turns out g, g', g'' are related to

$[X_{01}, X_{23}]$, $[X_{02}, X_{13}]$, $[X_{03}, X_{12}]$

LEM 69

u	$[X_{01}, X_{23}]$	$[X_{02}, X_{13}]$	$[X_{03}, X_{12}]$
gug^{-1}	$[X_{01}, X_{23}]$	$-[X_{02}, X_{13}]$	$-[X_{03}, X_{12}]$
$g'u(g')^{-1}$	$-[X_{01}, X_{23}]$	$[X_{02}, X_{13}]$	$-[X_{03}, X_{12}]$
$g''u(g'')^{-1}$	$-[X_{01}, X_{23}]$	$-[X_{02}, X_{13}]$	$[X_{03}, X_{12}]$

pf Use LEM 68 and the def of g, g', g'' $\square$

LEM 70 In the notation of Prop 64,

$$[X_{01}, X_{23}] : \qquad 4t \begin{pmatrix} 1 & -t^{-1} \\ 1 & -1 \end{pmatrix}$$

$$[X_{02}, X_{13}] : \qquad \frac{4}{1-t} \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

$$[X_{03}, X_{12}] : \qquad -\frac{4}{t} \begin{pmatrix} 1 & -1 \\ t & -1 \end{pmatrix}$$

pf Use Prop 64 $\square$

Comparing L67, L70, on $V(t)$

$[X_{01}, X_{23}]$ is a nonzero scalar mult of g

$[X_{02}, X_{13}]$ " " g'

$[X_{03}, X_{12}]$ " " g''

LEM 71 We have

$$g g' = -g' g$$

$$g g'' = -g'' g$$

$$g' g'' = -g'' g'$$

pf Use L69 and above comment. $\square$

By LEM 71

$$g g' g'' \quad \text{commutes with each of } g, g', g''$$

So we expect $g g' g'' \in \mathbb{C} I$.

LEM 72 Using the normalization of LEM 67,

$$g g' g'' = (t-1) I.$$

pf Matrix mult. $\square$

On $\mathbb{V}(t)$

$[X_{01}, X_{23}]$ is invertible and trace 0

So it has eigenvalues $\theta, -\theta$ $\quad 0 \neq \theta \in \mathbb{C}$

So

$$H := \frac{[X_{01}, X_{23}]}{\theta} \quad \text{has eigenvalues } 1, -1$$

H is defined up to sign, since we could replace θ by $-\theta$

We similarly define H', H''.

LEM 73 On $\mathbb{V}(t)$,

$$H \in \mathbb{C}g = \mathbb{C}[x_{01}, x_{23}]$$

$$H' \in \mathbb{C}g' = \mathbb{C}[x_{02}, x_{13}]$$

$$H'' \in \mathbb{C}g'' = \mathbb{C}[x_{03}, x_{12}]$$

Each of H, H', H'' has eigvals ± 1, $\det = -1$

$H^2 = I$, $(H')^2 = I$, $(H'')^2 = I$,

$HH' = -H'H$ $\quad$ $HH'' = -H''H$

$H'H'' = -H''H'$

$HH'H'' = \mp i I$ $\quad$ $i^2 = -1$

pf Consider last assertion. By L 72

$\exists\ \alpha \in \mathbb{C}$ s.t $HH'H'' = \alpha I$

Take the det of each side to get $-1 = \alpha^2$. $\square$

Thm 74 $\quad H, H', H''$ can be chosen such that

$$[H, H'] = 2i^{\circ} H''$$

$$[H', H''] = 2i^{\circ} H,$$

$$[H'', H] = 2i^{\circ} H'.$$

pf $\quad$ Replacing H by $-H$ if nec, wlog

$$HH'H'' = i^{\circ} I.$$

Also by L73

$$H'H''H = i^{\circ} I, \qquad H''HH' = i^{\circ} I$$

Obs
$$\begin{aligned}[H, H'] &= HH' - H'H \\ &= 2HH' \\ &= 2HH'H''H'' \\ &= 2i^{\circ} H''\end{aligned}$$

the other equations are similar. $\square$