Section 4. A combinatorial condition Kenneth Ma

Lemma 1. Let $\Gamma=(X,R)$ be a bipartite distance-regular graph with $D\ge 3$ and $k\ge 3$, then $p^i_{2i}>0$ $(1\le i\le D-1)$

Pf. Suppose $p^i_{2i}=0$. Then as $c_i, b_i, c_{i+1}, b_{i-1}$ are positive, we have $c_{i+1}=b_{i-1}=1$.
But $0<c_i\le c_{i+1}$, so $c_i=1$ and similarly $b_i=1$.
As $k=c_i+b_i$, $k=2$, which is absurd.

Theorem 2. Let $\Gamma=(X,R)$ be a bipartite DRG graph with $D\ge 3$ and $k\ge 3$, then

$$(b_{i-1}-1)(c_{i+1}-1)\ge(\mu-1)p^i_{2i} \qquad (1\le i\le D-1)$$

Pf: For given i and fixed $y,z\in X$, where $\partial(y,z)=i$, set $P=\Gamma_2(y)\cap\Gamma_i(z)$.
Note $|P|=p^i_{2i}>0$ by lemma 1. Take $x\in P$.
Moreover, set $\gamma(x)=|\Gamma_1(x)\cap\Gamma_1(y)\cap\Gamma_{i-1}(z)|$

$\delta(x)=|\Gamma_1(x)\cap\Gamma_1(y)\cap\Gamma_{i+1}(z)|$

Note $\gamma(x)+\delta(x)=c_2$ $(*)$

Claim $\left(\sum_{x\in P}\gamma(x)\right)\left(\sum_{x\in P}\delta(x)\right)\ge|P|\left(\sum_{x\in P}\gamma(x)\delta(x)\right)$ $(**)$

Obtained by expanding $\sum_{x\in P}(\gamma(x)-\bar\gamma)^2\ge 0$, where $\bar\gamma=|P|^{-1}\sum_{x\in P}\gamma(x)$ and $(*)$ and Cauchy inequality.

Counting the order pair wx where $w \in \Gamma_1(y) \cap \Gamma_{i-1}(z)$, $x \in \Gamma_i(w) \cap \Gamma_i(z)$ and $x \neq y$.
as there are c_i choices of w and $(b_{i-1}-1)$ choices of ~~b_{i-1}~~ $x\,(\neq y)$, we have $\#(wx) = c_i(b_{i-1}-1)$. But any element in $\sum_{x\in P} r(x)$ has an one-to-one correspondance with order pair wx, so $\sum_{x\in P} r(x) = c_i(b_{i-1}-1)$. Similarly, (1)

$\sum_{x\in P} s(x) = b_i(c_{i+1}-1)$ and (2)

$\sum_{x\in P} r(x)s(x) = c_i b_i(\cancel{p}\,c_2-1)$ by (3)

Counting order pair ~~x~~ uvx where $u \in \Gamma_1(y) \cap \Gamma_{i-1}(z)$, $v \in \Gamma_1(y) \cap \Gamma_{i+1}(z)$, and $x \in \Gamma_i(u) \cap \Gamma_i(v)$.

Now Plug (1) (2) (3) into (**), we have our desired result.

Thm Let $\Gamma=(X,R)$ be a ~~DRG~~ bipartite DRG with the same assumption with Thm 2. TFAE,

(i) $(b_{i-1}-1)(c_{i+1}-1) = (c_2-1)p^i_{2i}$

(ii) Exist $y, z \in X$ s.t. $\partial(y,z)=i$ and s.t. for all $x \in X$ with $\partial(x,y)=2$, $\partial(x,z)=i$, the number $|\Gamma_i(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ is independent of x.

(iii) Exist $x, y \in X$ s.t. $\partial(x,y)=2$ and s.t. for all $z \in X$ with $\partial(x,z)=i$, $\partial(y,z)=i$, the number $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ is independent of z.

(iv) For all $x, y, z \in X$ with $\partial(x,y)=2$, $\partial(x,z)=i$, $\partial(y,z)=i$

$$|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = \frac{c_i(b_{i-1}-1)}{p^i_{2i}}$$

(i) $\Rightarrow$ (iv) Take $\sum_{x \in P} (\gamma(x) - \bar{\gamma}) = 0$, we have $\gamma(x) = \bar{\gamma}$ for all $x \in P$

Computing $\bar{\gamma}$ with (1), we have (iv)

(iv) $\Rightarrow$ (ii) Trivial

(ii) $\Rightarrow$ (i) $\gamma(x)$ is independent of x, so $\gamma(x) = \bar{\gamma}$.

(i) $\Leftrightarrow$ (iii) Similar.