

Ivy Lin - Final Presentation

Finite dimensional irreducible $\mathfrak{A}$ -modules & Flags.

Previously

lem: $V = v.s. / K$ ^{K alg closed} $\dim \neq 0, < +\infty$. A, A^* (linear trans.
 $\Rightarrow \forall \lambda \in K$ T.F.A.E.

① $[A, [A, [A, A^*]]] - 4[A, A^*]$ vanishes on $V_A(\lambda)$

② $A^* V_A(\lambda) \subseteq V_A(\lambda+2) + V_A(\lambda) + V_A(\lambda-2)$

def of $\mathfrak{A}$, K , alg closed.

$V =$ positive f. dim irred $\mathfrak{A}$ -module

thm: $r, s, t, u \in \mathbb{I}$, $r \neq s, t \neq u$, $\forall \lambda \in K$.

$$X_{tu} V_{rs}(\lambda) \subseteq V_{rs}(\lambda+2) + V_{rs}(\lambda) + V_{rs}(\lambda-2)$$

where $V_{rs}(\lambda) := V_{X_{rs}}(\lambda)$.

① $t=r, u=s$	$(X_{tu} - \lambda I) V_{rs}(\lambda) = 0.$
② $t=s, u=r$	$(X_{tu} + \lambda I) V_{rs}(\lambda) = 0.$
③ $t=s, u \neq r$	$(X_{tu} + \lambda I) V_{rs}(\lambda) \subseteq V_{rs}(\lambda+2)$
④ $t \neq r, u=s$	$(X_{tu} - \lambda I) V_{rs}(\lambda) \subseteq V_{rs}(\lambda+2)$
⑤ $t=r, u \neq s$	$(X_{tu} - \lambda I) V_{rs}(\lambda) \subseteq V_{rs}(\lambda-2)$
⑥ $t \neq s, u=r$	$(X_{tu} + \lambda I) V_{rs}(\lambda) \subseteq V_{rs}(\lambda-2)$
⑦ r, s, t, u distinct.	$X_{tu} V_{rs}(\lambda) \subseteq V_{rs}(\lambda-2) + V_{rs}(\lambda) + V_{rs}(\lambda+2)$

lem: $V = v.s. / K$. A, B , linear tras / V .

$\Rightarrow \forall \lambda \in K$, T.F.A.E.

$\mathcal{D}[A|B] - 2A - 2B$ vanishes on $V_A(\lambda)$

$$\textcircled{2} (B + \lambda I)V_A(\lambda) \subseteq V_A(\lambda + 2)$$

$$\Psi = AB - BA - 2A - 2B.$$

$$\begin{aligned} \Rightarrow \forall v \in V_A(\lambda), \quad \Psi v &= ABv - BAv - 2Av - 2Bv \\ (Av = \lambda v) \quad &= A Bv - \lambda Bv - 2\lambda v - 2Bv \\ &= (A - (\lambda + 2)I)(B + \lambda I)v = 0. \end{aligned}$$

Next: $\dim V_{rs}(\lambda)$ is independent of rs .

thm: r, s, t distinct, $\forall \lambda \in K$.

$$V_{rs}(\lambda) + V_{rs}(\lambda - 2) + \dots = V_{rt}(\lambda) + V_{rt}(\lambda - 2) + \dots$$

$$\text{If: } \quad Y \quad \quad \quad Z.$$

$$\begin{aligned} Y \subseteq Z. \quad \exists j \text{ s.t. } Y &= \sum_{i=0}^j V_{rs}(\lambda - 2i) \\ Z &= \sum_{i=0}^j V_{rt}(\lambda - 2i). \end{aligned}$$

Z is the space on which

$$\prod_{i=0}^j (X_{rt} - (\lambda - 2i)I).$$

$$(X_{rt} - \lambda I) V_{rs}(\lambda) \subseteq V_{rs}(\lambda - 2)$$

$$\forall V_{rs}(\lambda - 2i) \xrightarrow{(X_{rt} - (\lambda - 2i)I)} V_{rs}(\lambda - 2i - 2) \xrightarrow{(X_{rt} - (\lambda - 2i - 2)I)} \dots \rightarrow$$

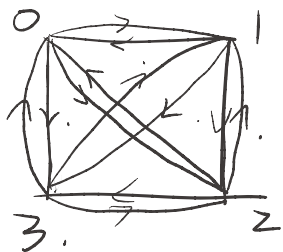
$$V(\lambda - 2j - 2).$$

$$\parallel$$

$$0.$$

(or: $\dim V_{rs}(\lambda)$ is independent of r, s .)

$$X_{rs} \sim X_{tu} : r=t \text{ or } s=u.$$



$$X_{rs} \sim X_{sr}.$$

$$\textcircled{1}. r=t.$$

$$V_{rs}(\lambda) + V_{rs}(\lambda - 2) + \dots = V_{rt}(\lambda) + V_{rt}(\lambda - 2) + \dots$$

$$V_{rs}(\lambda - 2) + \dots = V_{rt}(\lambda - 2) + \dots$$

$$H = H.$$

$$V_{rs}(\lambda) + H = V_{rt}(\lambda) + H \Rightarrow \dim V_{rs}(\lambda) = \dim V_{rt}(\lambda)$$

$$②. S=U.$$

$$\begin{aligned} \dim V_{rs}(\lambda) &= \dim V_{sr}(-\lambda) \\ &= \dim V_{ut}(-\lambda) \\ &= \dim V_{tu}(\lambda) \end{aligned}$$

$$\text{cor: } \dim V_{rs}(\lambda) = \dim V_{rs}(-\lambda). \quad \square$$

Goal: $V =$ positive f. dim irred $\mathbb{K}$ -module

① $\forall r, s \in \mathbb{I}, X_{rs}$ diagonalizable

② $\exists d \geq 0$ s.t. $\forall r, s \in \mathbb{I}$, eig.val. of X_{rs} are
 $\uparrow$
 $d, d-2, d-4, \dots, -d$

proof of main thm:

Let $\lambda =$ an eigval of X_{rs} .

$$\dots \lambda-4 \quad \lambda-2 \quad \lambda \quad \lambda+2 \quad \lambda+4 \dots$$

$\exists \theta \in \mathbb{K}$ s.t. θ is eigval, $\theta+2$ is not.

$\exists d \in \mathbb{N}$ s.t. $\theta-2d$ is eigval. $\theta-2d-2$ is not.

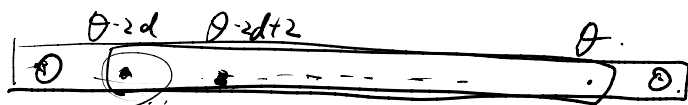
$$V_{rs}(\theta-2d) \dots V_{rs}(\theta) \neq 0,$$

$$V_{rs}(\theta-2d-2) = V_{rs}(\theta+2) = 0.$$

$$\text{Goal: } V_{rs}(\theta-2d) + \dots + V_{rs}(\theta) = V.$$

$\downarrow$
 W

$$X_{tu} W \subseteq W \Rightarrow W = V.$$



$$X_{tu} V_{rs}(\lambda) \subseteq V_{rs}(\lambda+2) + V_{rs}(\lambda) + V_{rs}(\lambda-2)$$

$$\theta = d \text{ by cor}$$

def: diameter of V is d .

def: flag on V of diameter d is

$$F_0, F_1, \dots, F_d, \text{ s.t.}$$

$$0 \neq F_0 \subseteq F_1 \subseteq \dots \subseteq F_d = V$$

call F_i the i th component

def: decomposition of V of diameter d is

$$V_0, V_1, \dots, V_d. \text{ s.t. } V = V_0 \oplus V_1 \oplus \dots \oplus V_d.$$

decomposition yields a flag, by

$$F_i := V_0 + \dots + V_i$$

F is induced by V

def: F, G flags on V , F, G are opposite

if $\exists V_0, \dots, V_d$ decomposition s.t.

F is induced by $V_0, \dots, V_d$

G — — — — — $V_d, \dots, V_0$

$$\Rightarrow V_i = F_i \cap G_{d-i}$$

V is induced by F, G