Lecture 9 Monday Sept 23

Morphisms, strict morphisms, isomorphisms

Given crystals B, B' with same root data

A map $\psi : B \to B'$

is a <u>crystal isomorphism</u> whenever

ψ is bijection

For $x \in B$,

$$\mathrm{wt}(x) = \mathrm{wt}(\psi(x))$$

$$\varphi_i(x) = \varphi_i(\psi(x)) \qquad i \in I$$

$$\varepsilon_i(x) = \varepsilon_i(\psi(x))$$

$$\psi\, e_i(x) = e_i\, \psi(x)$$

$$\psi\, f_i(x) = f_i\, \psi(x)$$

$$[\ \psi(\phi) = \phi\ ]$$

A map $\psi : B \to B'$

is a <u>strict crystal morphism</u> whenever it satisfies

all the above requirements except pos the bij requirement.

A map $\psi : B \to B' \cup \{\emptyset\}$

is a <u>crystal morphism</u> whenever

For $x \in B$ st $\psi(x) \in B'$

$$wt(x) = wt(\psi(x))$$
$$\varphi_i(x) = \varphi_i(\psi(x)) \qquad i \in I$$
$$\varepsilon_i(x) = \varepsilon_i(\psi(x))$$

For $x, y \in B$ s.t $\psi(x), \psi(y) \in B'$,

$$x \xleftarrow{i} y \qquad i \in I$$

implies

$$\psi(x) \xleftarrow{i} \psi(y)$$

For example

$$\psi : B \to B \cup \{\emptyset\}$$
$$x \to \emptyset$$

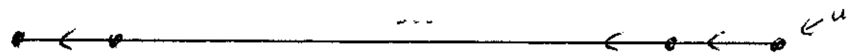
is a crystal morphism

Ex Fn $\Phi = A_1$, type $GL(2)$

Fix $k \geq 1$

Recall Crystals $B_{(k)}$, $T_{(k)}$, B_∞

$B_{(k)}$:

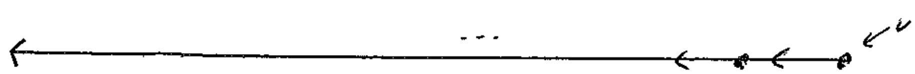


wt	ke_2		..	$(k-1)e_1+e_2$	ke_1
φ	0	1	..	$k-1$	k
ε	k	$k-1$	..	1	0

$T_{(k)}$:

wt	ke_1
φ	$-\infty$
ε	$-\infty$

B_∞



wt	..	e_2-e_1	0
φ	..	-1	0
ε	..	1	0

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Describe $T(k) \otimes B_\infty$

For $b \otimes b' \in T(k) \otimes B_\infty$

compare $\varepsilon(b)$ vs $\varphi(b')$

$$\underset{-\infty}{\overset{\|}{\varepsilon(b)}} \quad \text{vs} \quad \varphi(b') \in \mathbb{Z}$$

$$\varepsilon(b) < \varphi(b')$$

So

$$f(b \otimes b') = b \otimes f(b')$$

$$e(b \otimes b') = b \otimes e(b')$$

$$\varphi(b \otimes b') = \varphi(b') + \underbrace{\langle \underset{ke_1}{wt(b)}, \underset{e_1 - e_2}{\alpha_1^\vee} \rangle}_{k}$$

$$\varepsilon(b \otimes b') = \varepsilon(b')$$

$$wt(b \otimes b') = \underset{ke_1}{wt(b)} + wt(b')$$

$T_{(k)} \otimes B_\infty$:

wt	$\cdots$	$(k-1)e_1+e_2$	ke_1
φ	$\cdots$	$k-1$	k
ε	$\cdots$	1	0

The map

$$\Psi: \begin{array}{ccc} B_{(k)} & \longrightarrow & T_{(k)} \otimes B_\infty \\ f^i(u) & \longmapsto & t \otimes f^i(v) \end{array} \qquad 0 \le i \le k$$

is a crystal morphism that is not strict.

Next goal:

Given crystals B, C, D with same root data

show $(B \otimes C) \otimes D$ iso $B \otimes (C \otimes D)$

LEM Given crystals B, B' with same root data.

For $b \otimes b' \in B \otimes B'$ and $i \in I$.

(i) $\varphi_i(b \otimes b') = \max\{ \varphi_i(b), \; \varphi_i(b') + \langle wt(b), \alpha_i^\vee \rangle \}$

(ii) $\varepsilon_i(b \otimes b') = \max\{ \varepsilon_i(b) - \langle wt(b'), \alpha_i^\vee \rangle, \; \varepsilon_i(b') \}$

(iii)
$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varphi_i(b \otimes b') = \varphi_i(b) \\ b \otimes f_i(b') & \text{if } \varphi_i(b \otimes b') > \varphi_i(b) \end{cases}$$

(iv)
$$e_i(b \otimes b') = \begin{cases} e_i(b) \otimes b' & \text{if } \varepsilon_i(b \otimes b') > \varepsilon_i(b') \\ b \otimes e_i(b') & \text{if } \varepsilon_i(b \otimes b') = \varepsilon_i(b') \end{cases}$$

pf use table in def of $B \otimes B'$ $\square$

Prop Given crystals B, C, D with the same root data, the map

$$\Psi : \begin{array}{ccc} (B \otimes C) \otimes D & \longrightarrow & B \otimes (C \otimes D) \\ (b \otimes c) \otimes d & \longmapsto & b \otimes (c \otimes d) \end{array}$$

is a crystal isomorphism.

Pf For $x \in (B \otimes C) \otimes D$

write $x = (b \otimes c) \otimes d$

$$\begin{array}{l} wt(x) \overset{?}{=} wt(\Psi(x)) \\ \quad \| \\ \underbrace{wt(b \otimes c)}_{\substack{\| \\ wt(b) + wt(c)}} + wt(d) \end{array}$$

both sides equal

$wt(b) + wt(c) + wt(d)$

$$\varphi_i(x) \overset{?}{=} \varphi_i\big(\psi(x)\big) \qquad i \in I$$

$$\|$$

$$\max\Big\{ \underbrace{\varphi_i(b\otimes c)}_{\displaystyle \| \atop \displaystyle \max\{\varphi_i(b),\ \varphi_i(c) + \langle wt(b), \alpha_i^\vee\rangle\}},\ \ \varphi_i(d) + \langle \underset{\substack{\| \\ wt(b)+wt(c)}}{wt(b\otimes c)}, \alpha_i^\vee \rangle \Big\}$$

$$\|$$

$$\max\Big\{ \varphi_i(b),\ \ \varphi_i(c) + \langle wt(b), \alpha_i^\vee\rangle,\ \ \varphi_i(d) + \langle wt(b), \alpha_i^\vee\rangle + \langle wt(c), \alpha_i^\vee\rangle \Big\}$$

$$\varepsilon_i(x) \overset{?}{=} \varepsilon_i\big(\psi(x)\big)$$

$$\|$$

$$\max\Big\{ \underbrace{\varepsilon_i(b\otimes c)}_{\displaystyle \| \atop \displaystyle \max\{\varepsilon_i(b) - \langle wt(c), \alpha_i^\vee\rangle,\ \varepsilon_i(c)\}} - \langle wt(d), \alpha_i^\vee\rangle,\ \ \varepsilon_i(d) \Big\}$$

$$\|$$

$$\max\Big\{ \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee\rangle - \langle wt(d), \alpha_i^\vee\rangle,\ \ \varepsilon_i(c) - \langle wt(d), \alpha_i^\vee\rangle,\ \ \varepsilon_i(d) \Big\}$$

$$\psi\, e_i(x) \overset{?}{=} e_i\, \psi(x) \qquad i \in I$$

We have

$e_i((b \otimes c) \otimes d)$	$e_i(b \otimes (c \otimes d))$	Case
$(e_i(b) \otimes c) \otimes d$	$e_i(b) \otimes (c \otimes d)$	$\varepsilon_i(b \otimes c \otimes d) > \varepsilon_i(c \otimes d)$
$(b \otimes e_i(c)) \otimes d$	$b \otimes (e_i(c) \otimes d)$	$\varepsilon_i(b \otimes c \otimes d) = \varepsilon_i(c \otimes d) > \varepsilon_i(d)$
$(b \otimes c) \otimes e_i(d)$	$b \otimes (c \otimes e_i(d))$	$\varepsilon_i(b \otimes c \otimes d) = \varepsilon_i(c \otimes d) = \varepsilon_i(d)$

In each case

$$\psi\, e_i(x) = e_i\, \psi(x)$$

$$\Psi f_i(x) \overset{?}{=} f_i \Psi(x) \qquad i \in I$$

We have

$f_i((b\otimes c)\otimes d)$	$f_i(b\otimes(c\otimes d))$	Case
$(f_i(b)\otimes c)\otimes d$	$f_i(b)\otimes(c\otimes d)$	$\varphi_i(b\otimes c\otimes d) = \varphi_i(b\otimes c) = \varphi_i(b)$
$(b\otimes f_i(c))\otimes d$	$b\otimes(f_i(c)\otimes d)$	$\varphi_i(b\otimes c\otimes d) = \varphi_i(b\otimes c) > \varphi_i(b)$
$(b\otimes c)\otimes f_i(d)$	$b\otimes(c\otimes f_i(d))$	$\varphi_i(b\otimes c\otimes d) > \varphi_i(b\otimes c)$

In each case

$$\Psi f_i(x) = f_i(\Psi(x))$$

$\square$

From now on we identify

$$(B\otimes C)\otimes D, \qquad B\otimes(C\otimes D)$$

via Ψ, and write the result as

$$B\otimes C\otimes D$$

Prop For $k \geq 1$,

For crystals $B_1, B_2, \ldots, B_k$ with same root data,

For $b_1 \otimes \cdots \otimes b_k \in B_1 \otimes \cdots \otimes B_k$,

$$wt(b_1 \otimes \cdots \otimes b_k) = \sum_{l=1}^{k} wt(b_l)$$

For $i \in I$,

$$\varphi_i(b_1 \otimes \cdots \otimes b_k) =$$

$$\max \left\{ \varphi_i(b_j) + \sum_{l=1}^{j-1} \langle wt(b_l), \alpha_i^\vee \rangle \;\middle|\; 1 \leq j \leq k \right\}$$

$\Big[$ so $\varphi_i(b_1) \leq \varphi_i(b_1 \otimes b_2) \leq \cdots \leq \varphi_i(b_1 \otimes \cdots \otimes b_k)$ $\Big]$

$$f_i(b_1 \otimes \cdots \otimes b_k) = b_1 \otimes \cdots \otimes b_{t-1} \otimes f_i(b_t) \otimes b_{t+1} \otimes \cdots \otimes b_k$$

where

$$t = \min \left\{ j \;\middle|\; 1 \leq j \leq k,\ \varphi_i(b_1 \otimes \cdots \otimes b_j) = \varphi_i(b_1 \otimes \cdots \otimes b_k) \right\}$$

$$\varepsilon_i(b_1\otimes\cdots\otimes b_k) =$$

$$\max\left\{ \varepsilon_i(b_j) - \sum_{\ell=j+1}^{k} \langle wt(b_\ell), \alpha_i^\vee\rangle \;\middle|\; 1\le j\le k \right\}$$

$$\left[\text{so}\quad \varepsilon_i(b_1\otimes\cdots\otimes b_k) \ge \varepsilon_i(b_2\otimes\cdots\otimes b_k) \ge \cdots \ge \varepsilon_i(b_{k-1}\otimes b_k) \ge \varepsilon_i(b_k)\right]$$

$$e_i(b_1\otimes\cdots\otimes b_k) = b_1\otimes\cdots\otimes b_{s-1}\otimes e_i(b_s)\otimes b_{s+1}\otimes\cdots\otimes b_k$$

where $s = \max\left\{ j \;\middle|\; 1\le j\le k,\; \varepsilon_i(b_1\otimes\cdots\otimes b_k) = \varepsilon_i(b_j\otimes\cdots\otimes b_k) \right\}$

pf Use induction on k, viewing

$$\varphi_i(b_1\otimes\cdots\otimes b_k) = \varphi_i((b_1\otimes\cdots\otimes b_{k-1})\otimes b_k)$$

$$f_i(b_1\otimes\cdots\otimes b_k) = f_i((b_1\otimes\cdots\otimes b_{k-1})\otimes b_k)$$

$$\varepsilon_i(b_1\otimes\cdots\otimes b_k) = \varepsilon_i(b_1\otimes(b_2\otimes\cdots\otimes b_k))$$

$$e_i(b_1\otimes\cdots\otimes b_k) = e_i(b_1\otimes(b_2\otimes\cdots\otimes b_k))$$

$\square$