Lecture 8 Friday Sept 20 9/20/19

Given crystals B, B' with same root data

Φ, Λ, Σ

Their tensor product is the following crystal.

the vertex set is the Cartesian product of B and B', written

$$B \otimes B' = \{ b \otimes b' \mid b \in B, \ b' \in B' \}$$

For $b \otimes b' \in B \otimes B'$ and $i \in I$,

Case	$f_i(b\otimes b')$	$e_i(b\otimes b')$	$\varphi_i(b\otimes b')$	$\varepsilon_i(b\otimes b')$
$\varepsilon_i(b) > \varphi_i(b')$	$f_i(b)\otimes b'$	$e_i(b)\otimes b'$	$\varphi_i(b)$	$\varepsilon_i(b) - \varphi_i(b') + \varepsilon_i(b')$ $= \varepsilon_i(b) - \langle wt(b'), \alpha_i^\vee \rangle$
$\varepsilon_i(b) = \varphi_i(b')$	$f_i(b)\otimes b'$	$b \otimes e_i(b')$	$\varphi_i(b)$	$\varepsilon_i(b')$
$\varepsilon_i(b) < \varphi_i(b')$	$b \otimes f_i(b')$	$b \otimes e_i(b')$	$\varphi_i(b) + \varphi_i(b') - \varepsilon_i(b)$ $= \varphi_i(b') + \langle wt(b), \alpha_i^\vee \rangle$	$\varepsilon_i(b')$

where $x \otimes \phi = \phi = \phi \otimes y \quad \forall x \in B \ \forall y \in B'$

Also

$$wt(b \otimes b') = wt(b) + wt(b')$$

Check $B \otimes B'$ really is a crystal.

Check axioms A1, A2

A1. Given $x, y \in B \otimes B'$ and $i \in I$

Write $x = a \otimes a'$ $y = b \otimes b'$

Show $e_i(x) = y$ iff $f_i(y) = x$

Assume $f_i(y) = x$. show $e_i(x) = y$

Case $\varepsilon_i(b) \geq \varphi_i(b')$

$a \otimes a' = x = f_i(y) = f_i(b \otimes b') = f_i(b) \otimes b'$

$a \xleftarrow{i} b$

$a = f_i(b)$ and $a' = b'$

$e_i(a) = b$ and $\varepsilon_i(a) = \varepsilon_i(b) + 1 > \varphi_i(b') = \varphi_i(a')$

$e_i(x) = e_i(a \otimes a') = \underset{\substack{\shortparallel \\ b}}{e_i(a)} \otimes \underset{\substack{\shortparallel \\ b'}}{a'} = y$

Case $\varepsilon_i(b) < \varphi_i(b')$

$a \otimes a' = x = f_i(y) = f_i(b \otimes b') = b \otimes f_i(b')$

$a' \xleftarrow{i} b'$

$a = b$ and $a' = f_i(b')$

$e_i(a') = b'$ and $\varphi_i(a') = \varphi_i(b') - 1 \geq \varepsilon_i(b) = \varepsilon_i(a)$

$e_i(x) = e_i(a \otimes a') = \underset{\substack{\shortparallel \\ b}}{a} \otimes \underset{\substack{\shortparallel \\ b'}}{e_i(a')} = y$

Assume $e_i(x) = y$ show $f_i(y) = x$

Case $\varepsilon_i(a) \leq \varphi_i(a')$

$b \otimes b' = y = e_i(x) = e_i(a \otimes a') = a \otimes e_i(a')$

$a' \xleftarrow{i} b'$

$a = b$ and $e_i(a') = b'$

$f_i(b') = a'$ and $\varphi_i(b') = \varphi_i(a') + 1 > \varepsilon_i(a) = \varepsilon_i(b)$

$f_i(y) = f_i(b \otimes b') = \underset{a}{\underset{\|}{b}} \otimes \underset{a'}{\underset{\|}{f_i(b')}} = x$

Case $\varepsilon_i(a) > \varphi_i(a')$

$b \otimes b' = y = e_i(x) = e_i(a \otimes a') = e_i(a) \otimes a'$

$a \xleftarrow{i} b$

$e_i(a) = b$ and $a' = b'$

$f_i(b) = a$ and $\varepsilon_i(b) = \varepsilon_i(a) - 1 \geq \varphi_i(a') = \varphi_i(b')$

$f_i(y) = f_i(b \otimes b') = \underset{a}{\underset{\|}{f_i(b)}} \otimes \underset{a'}{\underset{\|}{b'}} = x$

We have shown

$$e_i(x) = y \quad \text{iff} \quad f_i(y) = x.$$

Assume these hold.

$$x \xleftarrow{\ i\ } y$$

$\exists$ two Cases

I $\quad a \xleftarrow{\ i\ } b \qquad a' = b'$.

$$\varphi_i(b) - \varphi_i(a) = 1, \qquad \varepsilon_i(a) - \varepsilon_i(b) = 1$$

$$\varepsilon_i(b) \geq \varphi_i(b'), \qquad \varepsilon_i(a) > \varphi_i(a')$$

$$wt(b) - wt(a) = \alpha_i$$

II $\quad a = b \qquad a' \xleftarrow{\ i\ } b'$.

$$\varphi_i(b') - \varphi_i(a') = 1, \qquad \varepsilon_i(a') - \varepsilon_i(b') = 1$$

$$\varepsilon_i(b) < \varphi_i(b') \qquad \varepsilon_i(a) \leq \varphi_i(a')$$

$$wt(b') - wt(a') = \alpha_i$$

9|20|19
6

Show

$$wt(y) - wt(x) \overset{?}{=} \alpha_i$$

$$wt(y) = wt(b) + wt(b')$$

$$wt(x) = wt(a) + wt(a')$$

ok in Cases I, II ✓

Show $\varphi_i(y) - \varphi_i(x) = 1$

Case I

$$\varphi_i(y) - \varphi_i(x) \overset{?}{=} 1$$

$$\varphi_i(y) = \varphi_i(b \otimes b') = \varphi_i(b)$$

$$\varphi_i(x) = \varphi_i(a \otimes a') = \varphi_i(a)$$

Case II

$$\varphi_i(y) - \varphi_i(x) \overset{?}{=} 1$$

$$\varphi_i(y) = \varphi_i(b \otimes b') = \underline{\varphi_i(b)} + \varphi_i(b') - \varepsilon_i\underline{\underline{(b)}}$$

$$\varphi_i(x) = \varphi_i(a \otimes a') = \underline{\varphi_i(a)} + \varphi_i(a') - \varepsilon_i\underline{\underline{(a)}}$$

show $\varepsilon_i(x) - \varepsilon_i(y) = 1$

Case I

$$\varepsilon_i(x) - \varepsilon_i(y) \overset{?}{=} 1$$

$$\varepsilon_i(x) = \varepsilon_i(a \otimes a') = \varepsilon_i(a) - \underline{\varphi_i(a')} + \underline{\underline{\varepsilon_i(a')}}$$

$$\varepsilon_i(y) = \varepsilon_i(b \otimes b') = \varepsilon_i(b) - \underline{\varphi_i(b')} + \underline{\underline{\varepsilon_i(b')}}$$

Case II

$$\varepsilon_i(x) - \varepsilon_i(y) \overset{?}{=} 1$$

$$\varepsilon_i(x) = \varepsilon_i(a \otimes a') = \varepsilon_i(a')$$

$$\varepsilon_i(y) = \varepsilon_i(b \otimes b') = \varepsilon_i(b'')$$

Axiom A2

For $\underset{b \otimes b'}{\overset{\shortparallel}{y}} \in B \otimes B'$ and $i \in I$

show

$$\varphi_i(y) - \varepsilon_i(y) = \langle wt(y), \alpha_i^\vee \rangle$$

$$LHS = \varphi_i(b) - \varepsilon_i(b) + \varphi_i(b') - \varepsilon_i(b')$$

in each case

$\varepsilon_i(b) > \varphi_i(b')$, $\varepsilon_i(b) = \varphi_i(b')$, $\varepsilon_i(b) < \varphi_i(b')$

$$RHS = \langle wt(b) + wt(b'), \alpha_i^\vee \rangle$$

$$= \varphi_i(b) - \varepsilon_i(b) + \varphi_i(b') - \varepsilon_i(b')$$

✓

For $y \in B \otimes B'$ and $i \in I$
$\quad \overset{\shortparallel}{b \otimes b'}$

when is

$$\varphi_i(y) = -\infty = \varepsilon_i(y)$$

Cases

	$\varphi_i(b') = -\infty = \varepsilon_i(b')$ $f_i(b') = \phi = e_i(b')$	$\varphi_i(b'),\ \varepsilon_i(b') \in \mathbb{Z}$
$\varphi_i(b) = -\infty = \varepsilon_i(b)$ $f_i(b) = \phi = e_i(b)$	$\varepsilon_i(b) = \varphi_i(b')$ $f_i(y) = \phi \otimes b' = \phi$ $e_i(y) = b \otimes \phi = \phi$ $\varphi_i(y) = \varphi_i(b) = -\infty$ $\varepsilon_i(y) = \varepsilon_i(b') = -\infty$	$\varepsilon_i(b) < \varphi_i(b')$ $f_i(y) = b \otimes f_i(b')$ $e_i(y) = b \otimes e_i(b')$ $\varphi_i(y) = \varphi_i(b') + \langle wt(b), \alpha_i^\vee \rangle \in \mathbb{Z}$ $\varepsilon_i(y) = \varepsilon_i(b') \in \mathbb{Z}$
$\varphi_i(b),\ \varepsilon_i(b) \in \mathbb{Z}$	$\varepsilon_i(b) > \varphi_i(b')$ $f_i(y) = f_i(b) \otimes b'$ $e_i(y) = e_i(b) \otimes b'$ $\varphi_i(y) = \varphi_i(b) \in \mathbb{Z}$ $\varepsilon_i(y) = \varepsilon_i(b) - \langle wt(b'), \alpha_i^\vee \rangle \in \mathbb{Z}$	$\varphi_i(y) \in \mathbb{Z}$ $\varepsilon_i(y) \in \mathbb{Z}$

So $\quad \varphi_i(y) = -\infty$ iff $\varepsilon_i(y) = -\infty$

and in this case

$$e_i(y) = \phi \qquad f_i(y) = \phi \qquad \checkmark$$

Ex $\quad \Phi = A_1 \quad$ type $\quad GL(2)$

For crystals $\mathcal{B} = \mathcal{B}_{(3)}$ and $\mathcal{B}' = \mathcal{B}_{(4)}$

Describe crystals $\mathcal{B} \otimes \mathcal{B}'$ and $\mathcal{B}' \otimes \mathcal{B}$

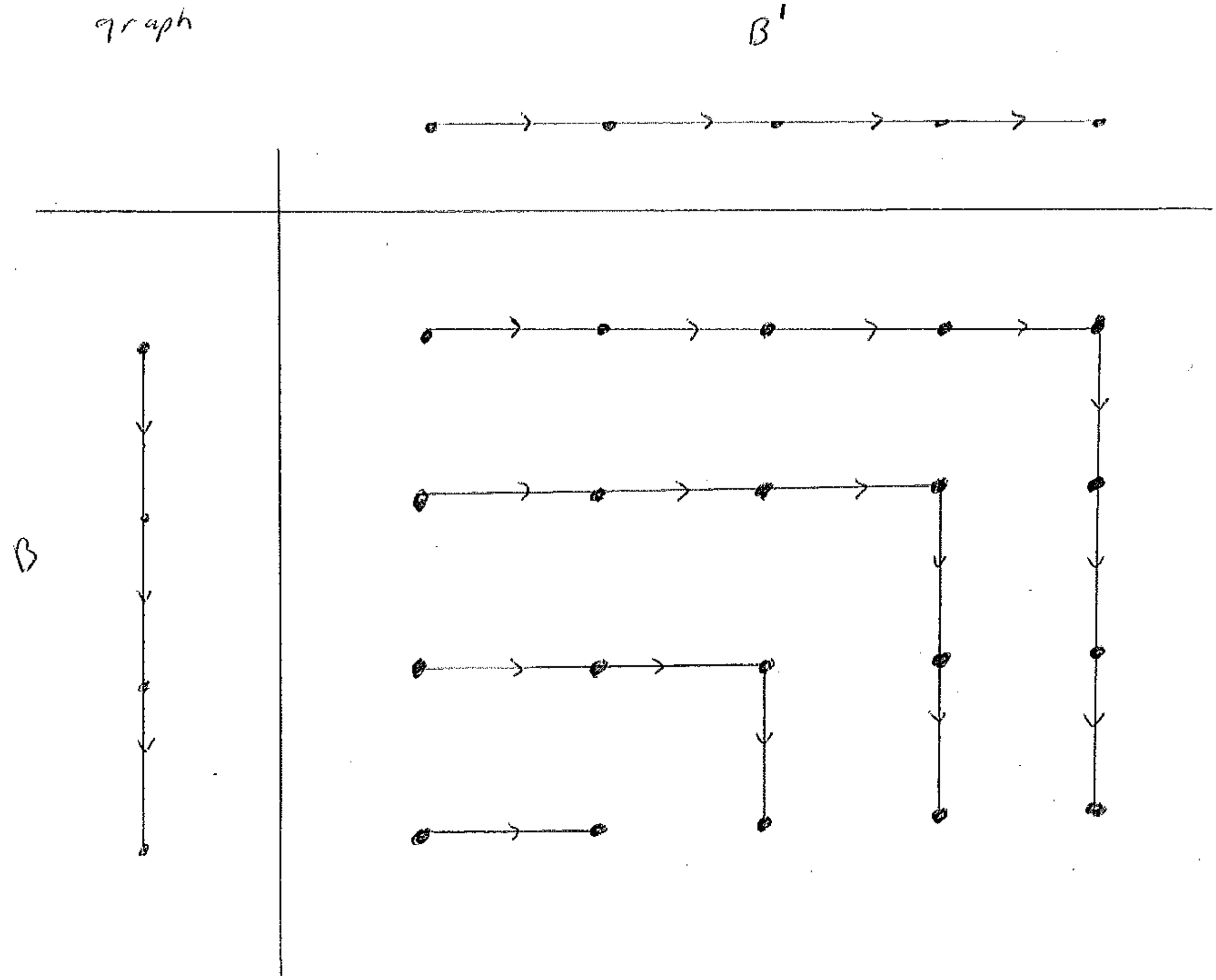
$\mathcal{B}$, $\mathcal{B}'$ are semi normal with crystal graphs

$\longrightarrow$ means $\xrightarrow[1]{}$

$\mathcal{B} \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$

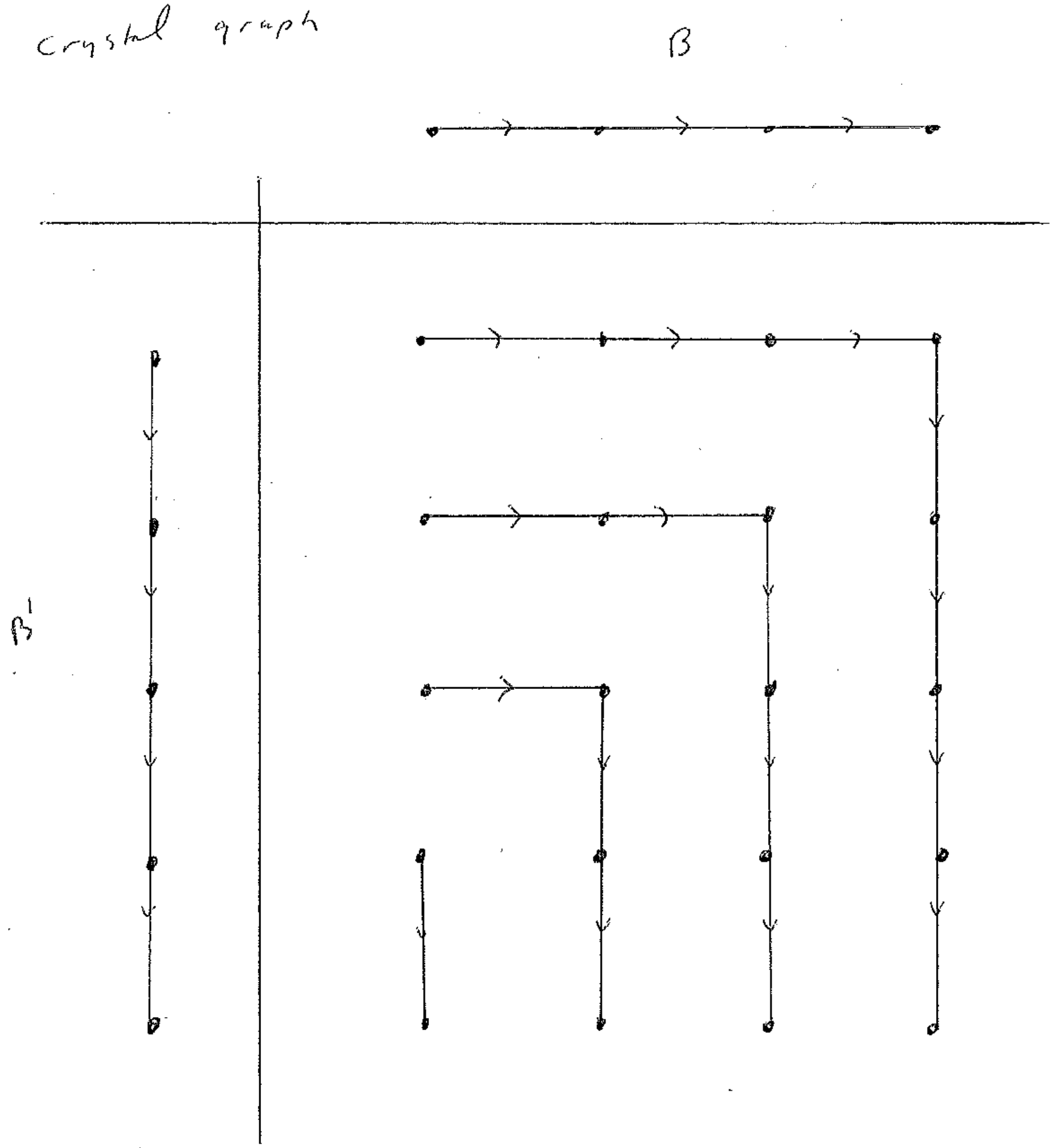
$\mathcal{B}' \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$

Crystal $B \otimes B'$ is semi normal with crystal graph



dot in row b, column b' represents $b \otimes b'$

Crystal $B' \otimes B$ is semi normal with crystal graph



Note $B \otimes B'$, $B' \otimes B$ are iso but the iso is <u>not</u>

$B \otimes B' \to B' \otimes B$

$b \otimes b' \to b' \otimes b$

LEM 15 Assume crystals B, B' are seminormal. then so is $B \otimes B'$.

pf Fix $x \in B \otimes B'$ and $i \in I$ show

(i) $\varphi_i(x) \geq 0$

(ii) $\varphi_i(x) = 0$ iff $f_i(x) = \phi$

(iii) $\varepsilon_i(x) \geq 0$

(iv) $\varepsilon_i(x) = 0$ iff $e_i(x) = \phi$

Write $x = b \otimes b'$

Consider table in def of $B \otimes B'$

(i), (iii) routine

(ii) One checks

$\varphi_i(b \otimes b') = 0$

iff $\varepsilon_i(b) \geq \varphi_i(b')$ and $\varphi_i(b) = 0$

iff $f_i(b \otimes b') = \phi$

(iv) one checks

$\varepsilon_i(b \otimes b') = 0$

iff $\varepsilon_i(b) \leq \varphi_i(b')$ and $\varepsilon_i(b') = 0$

iff $e_i(b \otimes b') = \phi$

$\square$

Ex Fn $\Phi = A_r$ type $GL(r+1)$

Fn crystal $\mathcal{B} = \mathcal{B}_{(1)}$

Find $\mathcal{B} \otimes \mathcal{B}$ for $r = 3$

Sol.

$$\mathcal{B} : \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \boxed{4}$$

Abbr. xy for $\boxed{x} \otimes \boxed{y}$

$\mathcal{B} \otimes \mathcal{B}$:

$$\begin{tikzcd}
11 \arrow[r, "1"'] & 12 \arrow[r, "2"'] \arrow[d, "1"] & 13 \arrow[r, "3"'] \arrow[d, "1"] & 14 \arrow[d, "1"] \\
21 \arrow[d, "2"] & 22 \arrow[r, "2"'] & 23 \arrow[r, "3"'] \arrow[d, "2"] & 24 \arrow[d, "2"] \\
31 \arrow[r, "1"'] \arrow[d, "3"] & 32 \arrow[d, "3"] & 33 \arrow[r, "3"'] & 34 \arrow[d, "3"] \\
41 \arrow[r, "1"'] & 42 \arrow[r, "2"'] & 43 & 44
\end{tikzcd}$$

$\xrightarrow[1]{}$

$\xrightarrow[2]{}$

$\xrightarrow[3]{}$