Lecture 7 Wednesday Sept 18

Ex For Φ type C_r

$\exists$ unique semi normal crystal with crystal graph and wts

$$\boxed{\bar{1}} \xleftarrow{1} \boxed{\bar{2}} \xleftarrow{2} \cdots \xleftarrow{r-1} \boxed{\bar{r}} \xleftarrow{r} \boxed{r} \xleftarrow{r-1} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

wt $\quad -e_1 \quad -e_2 \quad \cdots \quad -e_r \quad e_r \quad \cdots \quad e_2 \quad e_1$

this crystal is self dual

"standard crystal type C_r"

Ex For Φ type D_r

$\exists$ unique semi normal crystal with crystal graph and wts

$$\begin{tikzcd}
& & & & \boxed{r} \arrow[dl, "r"'] & & & & \\
\boxed{\bar{1}} & \boxed{\bar{2}} \arrow[l, "1"] & \cdots \arrow[l, "2"] & \boxed{\overline{r-1}} \arrow[l, "r-2"] & & \boxed{r-1} \arrow[ul, "r-1"'] \arrow[dl, "r"] & \cdots \arrow[l, "r-2"] & \boxed{2} \arrow[l, "2"] & \boxed{1} \arrow[l, "1"] \\
& & & & \boxed{\bar{r}} \arrow[ul, "r-1"] & & & &
\end{tikzcd}$$

$e_r \quad e_{r-1} \quad \cdots \quad e_2 \quad e_1$

wt: $-e_1 \quad -e_2 \quad \cdots \quad -e_{r-1} \quad -e_r$

This crystal is self dual

"standard crystal type D_r"

Ex For $\Phi = A_r$ type $GL(r+1)$

write $n = r+1$

For an integer $k \geq 1$ the Crystal $\mathcal{B} = \mathcal{B}_{(k)}$

is semi normal with vertex set

$$\boxed{a_1}\boxed{a_2}\boxed{\quad \cdots \quad}\boxed{a_k} \qquad a_i \in \{1, 2, \ldots, n\}$$

$$a_1 \leq a_2 \leq \cdots \leq a_k$$

For $x = \boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_k} \in \mathcal{B}$

- $$wt(x) = \sum_{i=1}^{k} wt\left(\boxed{a_i}\right)$$

where $wt\left(\boxed{j}\right) = e_j$

So $$wt(x) = \sum_{i=1}^{n} \lambda_i e_i$$

where $$\lambda_i = \left|\{ t \mid 1 \leq t \leq k,\ a_t = i \}\right|$$

- For $1 \le i \le r$ desc $e_i(x)$, $f_i(x)$

If none of $a_1 \dots a_k$ is i then

$$f_i(x) = \phi$$

Otherwise $f_i(x)$ is obtained from x by replacing rightmost i among $a_1 \dots a_k$ by $i+1$

If none of $a_1 \dots a_k$ is $i+1$ then

$$e_i(x) = \phi$$

otherwise $e_i(x)$ is obtained from x by replacing leftmost $i+1$ among $a_1, \dots, a_k$ by i

- For $1 \le i \le r$

$$\varphi_i(x) = \left| \left\{ t \mid 1 \le t \le k,\ a_t = i \right\} \right|$$

$$\varepsilon_i(x) = \left| \left\{ t \mid 1 \le t \le k,\ a_t = i+1 \right\} \right|$$

To illustrate, for $r=2$, $k=3$ the crystal graph is

$$\begin{tikzcd}
\boxed{222} \arrow[dr, "2"] & & \boxed{122} \arrow[ll, "1"'] \arrow[dr, "2"] & & \boxed{112} \arrow[ll, "1"'] \arrow[dr, "2"] & & \boxed{111} \arrow[ll, "1"'] \\
& \boxed{223} \arrow[dr, "2"] & & \boxed{123} \arrow[ll, "1"'] \arrow[dr, "2"] & & \boxed{113} \arrow[ll, "1"'] & \\
& & \boxed{233} \arrow[dr, "2"] & & \boxed{133} \arrow[ll, "1"'] & & \\
& & & \boxed{333} & & &
\end{tikzcd}$$

the wts are

$$\begin{array}{ccccccc}
030 & & 120 & & 210 & & 300 \\
& 021 & & 111 & & 201 & \\
& & 012 & & 102 & & \\
& & & 003 & & &
\end{array}$$

abc means $ae_1 + be_2 + ce_3$

For above crystal B, view the entries as "semistandard tableaux with one row"

Later, we define a crystal for any tableaux shape.

For now, consider tableaux with one column.

Ex For $\Phi = A_r$ type $GL(r+1)$ $n = r+1$

For $k \geq 1$ the crystal $B = B_{(1^k)}$ is semi normal with vertex set

$$\begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \qquad a_i \in \{1, 2, \dots, n\}$$

$$a_1 < a_2 < \dots < a_k$$

For $x = \begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \in B$

- $wt(x) = \sum_{i=1}^{k} wt\left(\boxed{a_i}\right)$

- Fn $\quad 1 \le i \le r \quad$ desc $\quad e_i(x), \quad f_i(x)$

If $a_1, \dots, a_k$ includes i but not $i+1$, replace i by $i+1$ to get $f_i(x)$. otherwise $f_i(x) = \phi$.

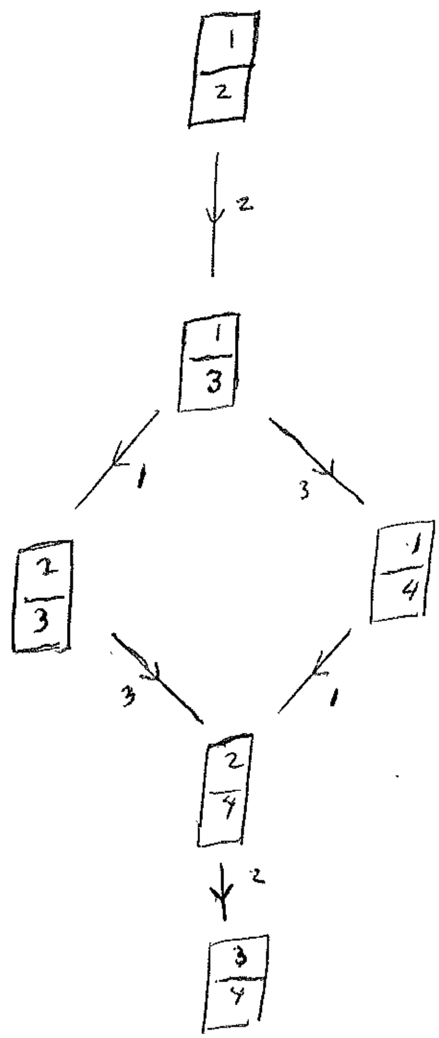
If $a_1 \dots a_k$ includes $i+1$ but not i, replace $i+1$ by i to get $e_i(x)$. otherwise $e_i(x) = \phi$.

- Fn $\quad 1 \le i \le r$

$$\varphi_i(x) = \begin{cases} 1 & \text{if } a_1 \dots a_k \text{ includes } i \text{ but not } i+1 \\ 0 & \text{else} \end{cases}$$

$$\varepsilon_i(x) = \begin{cases} 1 & \text{if } a_1 \dots a_k \text{ includes } i+1 \text{ but not } i \\ 0 & \text{else} \end{cases}$$

To illustrate, for $r = 3$ $k = 2$



wts

1100

1010

0110 1001

0101

0011

abcd means

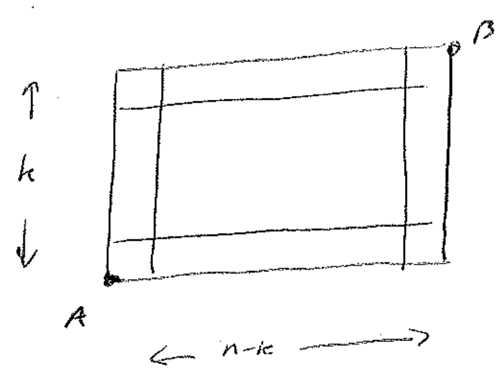
$ae_1 + be_2 + ce_3 + de_4$

We just drew a poset.

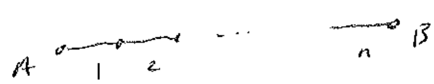
We now interpret the poset for general k, n

Elements of $\mathcal{B}$ are in bijection with k-subsets of $\{1, 2, \ldots, n\}$

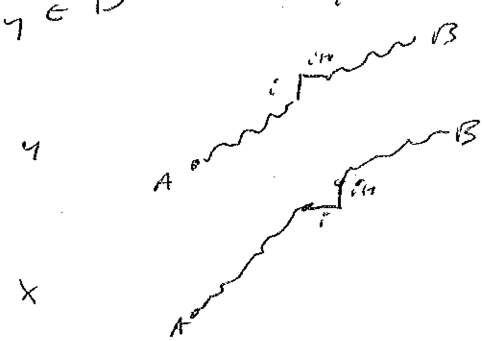
These k-subsets are in bijection with the Lattice paths of length n from A to B below:



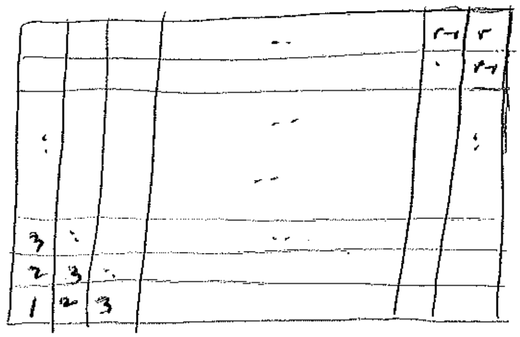
For an n-path label edges



For $x, y \in \mathcal{B}$ $\quad x \leftarrow_i y$ whenever



Label the Lattice blocks $1, 2, \ldots, r$ as follows

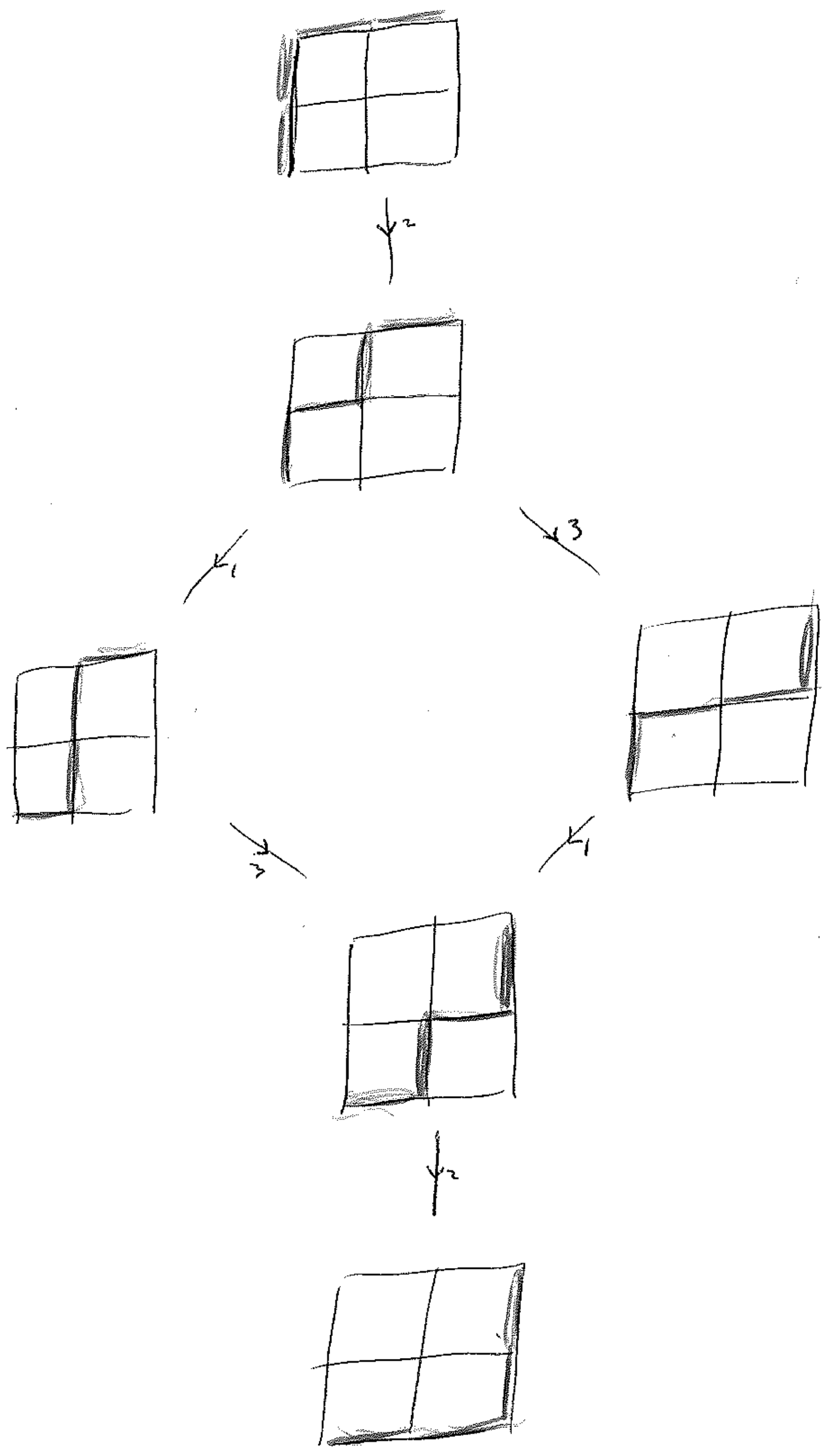


$x \xleftarrow[i]{} y$ whenever x is obtained from y by "adding" an i-block

$r = 3,\quad k = 2$ revisited

2	3
1	2

$r=3, \quad k=2 \qquad$ revisited



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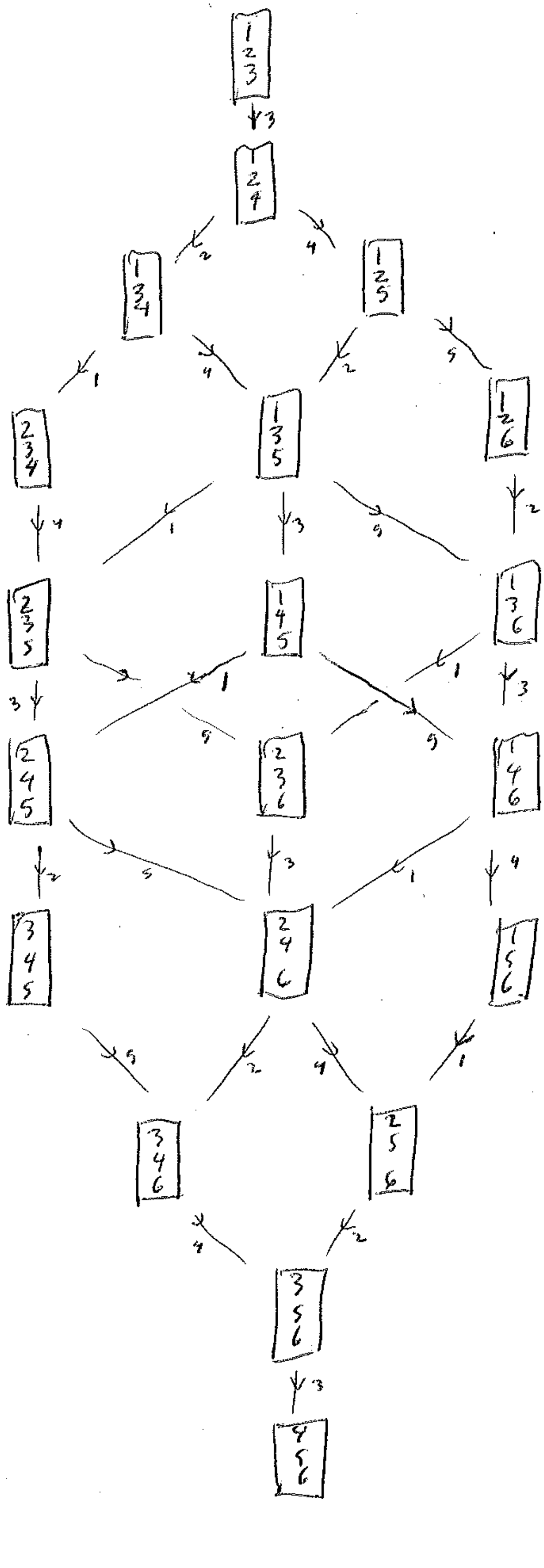
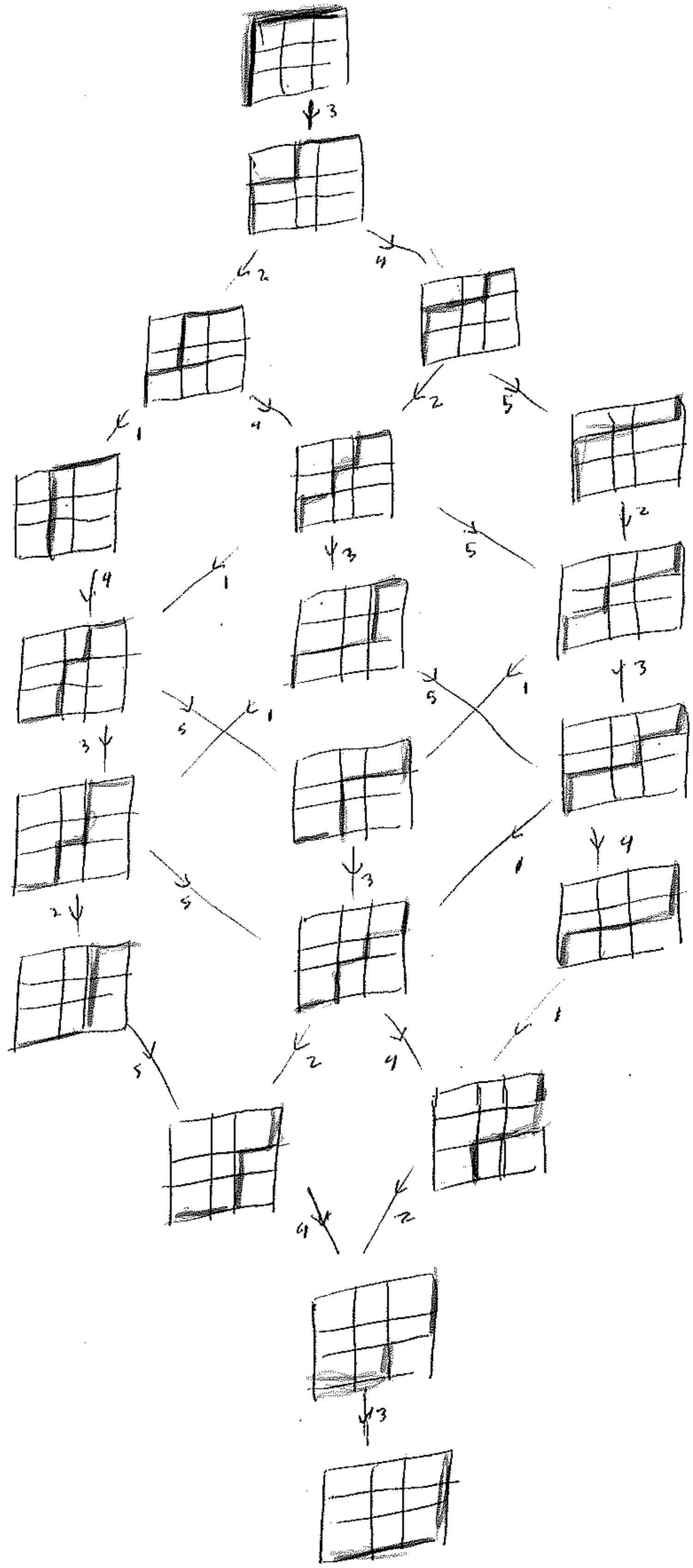
Take $k = 3$
$r = 5$





3	4	5
2	3	4
1	2	3

2 examples of a crystal that is not semi normal.

Ex For $\Phi = A_1$ type $GL(2)$.

the crystal B_∞ has crystal graph

and

$$\cdots \xleftarrow{1} v_{-4} \xleftarrow{1} v_{-2} \xleftarrow{1} v_0$$

	$2e_2-2e_1$	e_2-e_1	0	wt
$\cdots$	-2	-1	0	φ_1
$\cdots$	2	1	0	ε_1

Ex For any Φ, Λ and any $\lambda \in \Lambda$
crystal T_λ has unique vertex t_λ

and $e_i(t_\lambda) = \phi$ $\quad f_i(t_\lambda) = \phi$, $\quad i \in I$

$wt(t_\lambda) = \lambda$,

$\varphi_i(t_\lambda) = -\infty$, $\quad \varepsilon_i(t_\lambda) = -\infty$