Lecture 6 Monday Sept 16

B is <u>connected</u> whenever it has 1 connected component

An element $x \in B$ is <u>highest wt</u> whenever

$e_i(x) = \phi \quad \forall i \in I$

In this case $wt(x)$ is called a <u>highest wt</u> for B.

LEM 12 For a crystal B and $x \in B$ with $wt(x)$ maximal wrt partial order $\preceq$

[ie $\nexists\, y \in B$ st $wt(x) \prec wt(y)$]

then x is highest wt.

pf Suppose not. Then $\exists\, y \in B$ st $e_i(x) = y$

then $wt(y) - wt(x) = \alpha_i$

So $wt(x) \prec wt(y)$ contr

$\square$

LEM B Assume Crystal B is seminormal.

For a hw element $x \in B$,

$wt(x)$ is dominant.

pf for $i \in I$ show

$$\langle wt(x), \alpha_i^\vee \rangle \geq 0$$

By constr

$$\langle wt(x), \alpha_i^\vee \rangle = \varphi_i(x) - \varepsilon_i(x)$$

$\varphi_i(x) \geq 0$ by SN

$\varepsilon_i(x) = 0$ since $e_i(x) = \phi$

Result follows

$\square$

LEM14 Assume crystal $\mathcal{B}$ is seminormal.

Given $\lambda, \mu \in \Lambda$ and $w \in W$ s.t

$w(\lambda) = \mu$.

then

$\#$ elements in $\mathcal{B}$ of wt λ

$= \#$ elements in $\mathcal{B}$ of wt μ

pf W is gen by simple reflections.

WLOG $w =$ simple reflection. s_i

$$\mu = s_i(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha_i^\vee \rangle}_{= r \in \mathbb{Z}} \alpha_i$$

Swapping λ, μ if nec, WLOG $r \geq 0$

Define

$$\mathcal{B}_\lambda = \{ x \in \mathcal{B} \mid wt(x) = \lambda \}$$

$$\mathcal{B}_\mu = \{ y \in \mathcal{B} \mid wt(y) = \mu \}$$

show $|\mathcal{B}_\lambda| = |\mathcal{B}_\mu|$

For $x \in B_\lambda$

$$r = \langle \lambda, \alpha_i^\vee \rangle = \varphi_i(x) - \underset{\substack{\vee\! | \\ 0}}{\varepsilon_i(x)}$$

So

$$r \leq \varphi_i(x)$$

$$f_i^r(x) \in B$$

$f_i^r(x)$ has wt $\lambda - r\alpha_i = \mu$

$$f_i^r(x) \in B_\mu$$

Similary for $y \in B_\mu$

$$e_i^r(y) \in B_\lambda$$

Consider the functions

$$\begin{array}{ccc} B_\lambda & \longrightarrow & B_\mu \\ x & \longrightarrow & f_i^r(x) \end{array} \qquad *$$

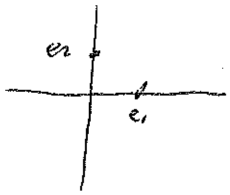
$$\begin{array}{ccc} B_\lambda & \longleftarrow & B_\mu \\ e_i^r(y) & \longleftarrow & y \end{array} \qquad **$$

$*$, $**$ are inverses, hence bijections.

So $|B_\lambda| = |B_\mu|$ $\square$

Ex $\quad \Phi = A_1 = \{\pm\alpha\} \quad \alpha = e_1 - e_2$

GL(2) type $\quad \Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$



For $k \in \mathbb{N}$ $\exists$ crystal $\mathcal{B} = \mathcal{B}(k)$ with

$k+1$ vertices $\{x_{k-2i}\}_{i=0}^{k}$ and

$e(x_\ell) = x_{\ell+2} \quad (\ell \neq k) \quad e(x_k) = \emptyset \qquad e = e_1$

$f(x_\ell) = x_{\ell-2} \quad (\ell \neq -k) \quad f(x_{-k}) = \emptyset \qquad f = f_1$

$\varphi(x_\ell) = \frac{k+\ell}{2} \qquad \varphi = \varphi_1$

$\varepsilon(x_\ell) = \frac{k-\ell}{2} \qquad \varepsilon = \varepsilon_1$

$wt(x_\ell) = \frac{k+\ell}{2} e_1 + \frac{k-\ell}{2} e_2$

The crystal graph is a directed path

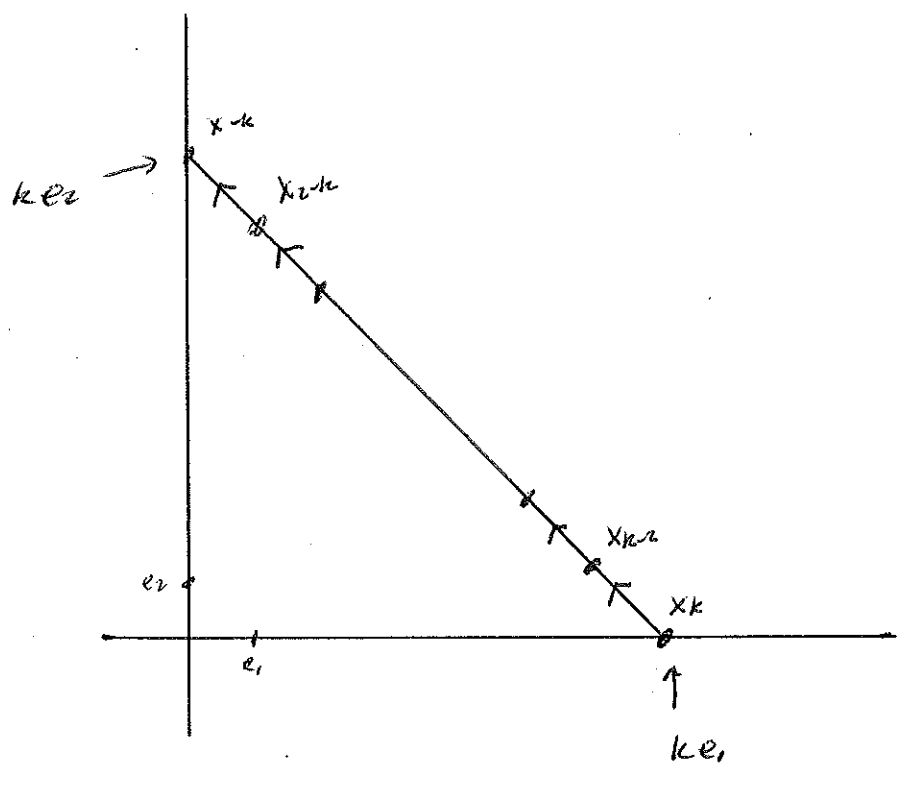
$x_{-k} \leftarrow x_{2-k} \leftarrow \cdots \leftarrow x_{k-2} \leftarrow x_k$

$\leftarrow$ means $\xleftarrow{1}$

	x_{-k}	x_{2-k}		$\cdots$	x_{k-2}	x_k
					$k-1$	k
φ	0	1	2	$\cdots$	1	0
ε	k	$k-1$	$k-2$	$\cdots$		

$\mathcal{B}$ is connected and semi normal.

We draw the crystal graph again, this time placing each vertex x at "location" $wt(x)$



B has unique highest wt element x_k
with highest wt ke_1

<u>Note</u> Above B is not the only connected
seminormal crystal with degree $k+1$.
The others are obtained from B by adding
an arb integer multiple of $\delta = e_1 + e_2$ to the
wt function for B.

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Ex $\Phi = A_r$

$GL(r+1)$ type $\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{r+1}$

$\exists$ unique seminormal crystal $\mathcal{B} = \mathcal{B}_{(1)}$

with vertices

$\boxed{i}$ $\quad i = 1, 2, \ldots, r+1$

and crystal graph

$$\boxed{r+1} \xleftarrow{r} \boxed{r} \longleftarrow \cdots \longleftarrow \boxed{3} \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

and $\mathrm{wt}\left(\boxed{i}\right) = e_i \qquad 1 \le i \le r+1$

"standard crystal of type $GL(r+1)$"

DEF For a crystal $\mathcal{B}$ with root data Φ, Λ, $\Sigma = \{\alpha_i\}_{i \in I}$

The set
$$\mathcal{B}^\vee = \{ x^\vee \mid x \in \mathcal{B} \}$$
becomes a crystal with the same root data, such that

(i) For $x, y \in \mathcal{B}$ and $i \in I$

$$x^\vee \xrightarrow[i]{} y^\vee$$

iff

$$x \xleftarrow[i]{} y$$

(ii) For $x \in \mathcal{B}$ and $i \in I$

$$\varphi_i(x^\vee) = \varepsilon_i(x)$$
$$\varepsilon_i(x^\vee) = \varphi_i(x)$$

(iii) For $x \in \mathcal{B}$,

$$wt(x^\vee) = -wt(x)$$

The crystal $\mathcal{B}^\vee$ is <u>dual</u> or <u>contragredient</u> to $\mathcal{B}$.

Ex Recall the standard crystal of type $GL(r+1)$ is seminormal with

$$\boxed{r+1} \xleftarrow{r} \boxed{r} \leftarrow \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

wt: e_{r+1} $\quad e_r \quad \cdots \quad e_2 \quad e_1$

The dual crystal is seminormal with

$$\boxed{r+1}^\vee \xrightarrow{r} \boxed{r}^\vee \longrightarrow \cdots \xrightarrow{2} \boxed{2}^\vee \xrightarrow{1} \boxed{1}^\vee$$

wt: $-e_{r+1} \quad -e_r \quad \cdots \quad -e_2 \quad -e_1$

For notational convenience abbreviate

$$\boxed{\bar{i}} = \boxed{i}^\vee$$

Dual crystal becomes

$$\boxed{\overline{r+1}} \xrightarrow{r} \boxed{\bar{r}} \longrightarrow \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

wt $-e_{r+1} \quad -e_r \quad \cdots \quad -e_2 \quad -e_1$

"dual standard crystal of type $GL(r+1)$"

Ex For Φ type B_r

$\exists$ unique semi normal crystal with crystal graph and wts

$$\boxed{\bar 1} \xleftarrow{1} \boxed{\bar 2} \xleftarrow{2} \cdots \xleftarrow{r-1} \boxed{\bar r} \xleftarrow{r} \boxed{0} \xleftarrow{r} \boxed{r} \xleftarrow{r-1} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

$$\text{wt} \quad -e_1 \quad -e_2 \quad \cdots \quad -e_r \quad 0 \quad e_r \quad \cdots \quad e_2 \quad e_1$$

This crystal is self dual

"standard crystal of type B_r"