Lecture 5 Friday Sept 13

B_r type $SO(r+1)$

Same as previous example, except take

$$\Lambda = \mathbb{Z}\Phi = \sum_{i=1}^{r} \mathbb{Z}e_i$$

C_r type $Sp(2r)$ "Symplectic"

$V = \mathbb{R}^r$

$\Phi = \{ \pm e_i \pm e_j \mid 1 \le i < j \le r \} \cup \{ \pm 2e_i \mid 1 \le i \le r \}$

This is co-root system for B_r

Φ is semisimple, not simply laced

Take

$\alpha_i = e_i - e_{i+1}$ $\quad 1 \le i \le r-1$

$\alpha_r = 2e_r$

$\Phi^+ = \{ e_i \pm e_j \mid 1 \le i < j \le r \} \cup \{ 2e_i \mid 1 \le i \le r \}$

$\mathbb{Z}\Phi = \{ \sum_{i=1}^r a_i e_i \mid a_i \in \mathbb{Z},\ a_1 + \dots + a_r \text{ even} \}$

Take

$\Lambda = \sum_{i=1}^r \mathbb{Z} e_i$

$= \mathbb{Z}\Phi \cup \{ \sum_{i=1}^r a_i e_i \mid a_i \in \mathbb{Z},\ a_1 + \dots + a_r \text{ odd} \}$

$\mathbb{Z}\Phi$ index 2 in Λ

Weyl group W same as for B_r

For $1 \le i \le r$

$$\varpi_i = e_1 + \dots + e_i$$

$$\Lambda_{sc} = \sum_{i=1}^{r} \mathbb{Z}\varpi_i = \sum_{i=1}^{r} \mathbb{Z}e_i = \Lambda$$

Weyl vector

$$\rho = \sum_{i=1}^{r} (r-k+1)\, e_i$$

For $\lambda = \sum_{i=1}^{r} \lambda_i e_i \in \Lambda$

$\lambda \geq 0$ iff $\lambda_1 + \dots + \lambda_i \geq 0 \qquad 1 \le i \le r$

$\lambda \in \Lambda^+$ iff $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$

D_r type spin(2r)

$V = \mathbb{R}^r$

$\Phi = \{ \pm e_i \pm e_j \mid 1 \le i < j \le r \}$

Φ semi simple

Fn $\alpha \in \Phi$ $\qquad \alpha^\vee = \alpha$

$\langle \alpha, \alpha \rangle = 2,$

Φ simply laced

Take

$\alpha_i = e_i - e_{i+1} \qquad 1 \le i \le r-1$

$\alpha_r = e_{r-1} + e_r$

$\Phi^+ = \{ e_i \pm e_j \mid 1 \le i < j \le r \}$

$\mathbb{Z}\Phi = \left\{ \sum_{i=1}^{r} a_i e_i \;\middle|\; a_i \in \mathbb{Z}, \quad a_1 + a_2 + \cdots + a_r \text{ even} \right\}$

Take

$$\Lambda = \underbrace{\sum_{i=1}^{r} \mathbb{Z} e_i}_{\text{"orthog wts"}} \cup \underbrace{\left\{ \frac{1}{2} \sum_{i=1}^{r} a_i e_i \,\middle|\, a_i \in \mathbb{Z}_{odd} \quad 1 \le i \le r \right\}}_{\text{"spin wts"}}$$

$$= \left\{ \frac{1}{2} \sum_{i=1}^{r} a_i e_i \,\middle|\, a_1, \dots, a_r \text{ integers with same parity} \right\}$$

We have

$$\mathbb{Z}\Phi \underset{\text{index } 2}{\subseteq} \sum_{i=1}^{r} \mathbb{Z} e_i \underset{\text{index } 2}{\subseteq} \Lambda$$

$$W = (\mathbb{Z}/2\mathbb{Z})^{r-1} \rtimes S_r$$

acts on $\{ \pm e_1, \pm e_2, \dots, \pm e_r \}$ by permutations and an even number of sign changes

For $1 \le i \le r$

$$\overline{\omega}_i = \begin{cases} e_1 + e_2 + \cdots + e_i & i \le r-2 \\ \dfrac{e_1 + \cdots + e_{r-1} - e_r}{2} & i = r-1 \\ \dfrac{e_1 + \cdots + e_{r-1} + e_r}{2} & i = r \end{cases}$$

$$\Lambda_{sc} = \sum_{i=1}^{r} \mathbb{Z}\bar{\omega}_i = \Lambda$$

Weyl vector

$$\rho = \sum_{i=1}^{r} (r-i) e_i$$

For $\lambda = \sum_{i=1}^{r} \lambda_i e_i \in \Lambda,$

$\lambda \gtrsim 0$ iff

$$\lambda_1 + \cdots + \lambda_i \geq 0 \qquad 1 \leq i \leq r-2,$$

$$\lambda_1 + \cdots + \lambda_{r-1} \geq |\lambda_r|$$

$\lambda \in \Lambda^+$ iff

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|$$

D_r type $SO(2r)$

Same as previous example, except take

$$\Lambda = \sum_{i=1}^{r} \mathbb{Z} e_i$$

Sporadic cases

E_6, E_7, E_8, F_4, G_2.

See Carter Lie algebras of finite and affine type

Kashiwara Crystals

Convention: for the set $\mathbb{Z} \cup \{-\infty\}$

understood

$$-\infty < n \qquad \forall n \in \mathbb{Z}$$

$$-\infty + n = -\infty \qquad \forall n \in \mathbb{Z}$$

Given a root system Φ with wt lattice Λ

Given simple roots $\Sigma = \{\alpha_i\}_{i \in I}$

A Kashiwara Crystal for above data is a nonempty set $\mathcal{B}$ together with functions

$$e_i, f_i : \mathcal{B} \longrightarrow \mathcal{B} \cup \{0\} \qquad i \in I$$

$$\varepsilon_i, \varphi_i : \mathcal{B} \longrightarrow \mathbb{Z} \cup \{-\infty\} \qquad i \in I$$

$$\mathrm{wt} : \mathcal{B} \longrightarrow \Lambda$$

such that

(A1) for $x, y \in B$ and $i \in I$

$$e_i(x) = y \quad \text{iff} \quad f_i(y) = x$$

In this case

$$wt(y) - wt(x) = \alpha_i$$

$$\varphi_i(y) - \varphi_i(x) = 1,$$

$$\varepsilon_i(x) - \varepsilon_i(y) = 1$$

(A2) For $x \in B$ and $i \in I$,

$$\varphi_i(x) - \varepsilon_i(x) = \left\langle wt(x), \alpha_i^\vee \right\rangle$$

Obs $\varphi_i(x) = -\infty$ iff $\varepsilon_i(x) = -\infty$. In this case require $e_i(x) = \phi$, $f_i(x) = \phi$

Note ϕ is a special symbol "auxiliary element" not contained in B

Names

$e_i, \ f_i$ Kashiwara operators

$\varepsilon_i, \ \varphi_i$ string length functions

wt weight map

$|B|$ degree

B has <u>finite type</u> whenever

$\varphi_i(x) \neq -\infty$
$\varepsilon_i(x) \neq -\infty$ $\forall \ x \in B \ \ \forall i \in I$

B is <u>seminormal</u> whenever for all $x \in B$ and $i \in I$

$$\varphi_i(x) = \max\{ t \geq 0 \mid f_i^t(x) \neq \phi \}$$

$$\varepsilon_i(x) = \max\{ t \geq 0 \mid e_i^t(x) \neq \phi \}$$

In this case B is finite type and

$$\varphi_i, \ \varepsilon_i : B \to \mathbb{N}$$

LEM 12 Assume Φ is semisimple.

B is a crystal of finite type, then

for $x \in B$

$$wt(x) = \sum_{i \in I} \left(\varphi_i(x) - \varepsilon_i(x) \right) \varpi_i$$

pf Φ spans V

$\{\alpha_i^\vee\}_{i \in I}$ basis for V

$\{\varpi_i\}_{i \in I}$ is dual basis for V rel $\langle , \rangle$

For $\lambda = wt(x)$

$$\lambda = \sum_{i \in I} \underbrace{\langle \lambda, \alpha_i^\vee \rangle}_{\overset{A2}{=} \ \varphi_i(x) - \varepsilon_i(x)} \varpi_i$$

$\square$

For a crystal B the corresponding <u>crystal graph</u> has vertex set B.

For $x, y \in B$ and $i \in I$, write

$$x \xrightarrow[i]{} y$$

iff $e_i(x) = y$

iff $f_i(y) = x$

Define a binary relation $\sim$ on B such that for $x, y \in B$ $x \sim y$ whenever $\exists$ path from x to y in the underlying undirected unlabelled graph.

$\sim$ is an equiv relation

Each equiv class of $\sim$ is a <u>connected component</u> of B

Any nonempty union of connected components inherits a crystal graph structure from B.

"full subcrystal"