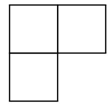
Recall that $\mathcal{B}_\lambda^{(n)}$ is the crystal of tableaux with highest weight λ. For example, $\mathcal{B}_(2,1)^{(n)}$ is the crystal of semi-standard tableaux of shape  whose entries lie in $1, 2, \ldots, n$.

The highest weight element is $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Our goal for the rest of the lecture today is to prove and show examples for the following:

Theorem 9.5(*The Littlewood-Richardson Rule*) *The multiplicity of $\mathcal{B}_\lambda^{(n)}$ in $\mathcal{B}_\mu^{(n)} \otimes \mathcal{B}_\nu^{(n)}$ equals $c_{\mu,\nu}^\lambda$.*

By Theorem 8.14, the multiplicity of $\mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal obtained from branching $\mathcal{B}_\lambda^{(r+s)}$ is $c_{\mu,\nu}^\lambda$. The result follows once we prove Theorem 9.4 below.

Theorem 9.4 *Let $\lambda, \mu,$, and ν be partitions. Then, the multiplicity of $\mathcal{B}_\lambda^{(n)}$ in $\mathcal{B}_\mu^{(n)} \otimes \mathcal{B}_\nu^{(n)}$ equals the multiplicity of $\mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal obtained by branching $B_\lambda^{(r+s)}$.*

To be clear, by multiplicity of $\mathcal{B}_\lambda^{(n)}$ in $\mathcal{B}_\mu^{(n)} \otimes \mathcal{B}_\nu^{(n)}$, we mean the number of connected components of the latter crystal that are isomorphic to the former.
Recall the following:

Theorem 8.6 *For $x \in \mathbb{B}^{\otimes k}$, $x \equiv P(x)$. Furthermore, if $P(x) = P(y)$, then $x \equiv y$.*

Theorem 8.7 *Let $x, y \in \mathbb{B}^{\otimes k}$. Then, $Q(x) = Q(y)$ if and only if x and y lie in the same connected component of $\mathbb{B}^{\otimes k}$.*

Now, we are ready to prove Theroem 9.4.

Proof of Theorem 9.4. Let

$$C := \{X \in \mathrm{Mat}_{(r+s)\times n)}(\mathbb{N}) | P(X) \in \mathcal{B}_\lambda^{(n)}, Q'(X) \in \mathcal{B}_\mu^{(r)}, Q''(X) \in \mathcal{B}_\nu^{(s)}\}.$$

By Theorems 8.6 and 8.7, C consists of every element of $\mathrm{Mat}_{(r+s)\times n}(\mathbb{N})$ that are $\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)-$ plactically equivalent to some element of the $\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal $\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$. Because C consists of all such elements, it must be a disjoint union of copies of the crystal $\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$. We will prove the theorem by counting the number of copies of this crystal in C.

First, we observe that

$$C \subset C_1 := \{X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) | P(X) \in \mathcal{B}_\lambda^{(n)}\}.$$

We note that the isomorphism of Corollary 9.2 is given by $X \mapsto P(X) \boxtimes Q(X)$. Hence, we have that $C_1 \cong \mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\lambda^{(r+s)}$. As discussed last Friday (lecture 40, part 2), we can branch (using Levi branching) the $\mathrm{GL}(n) \times \mathrm{GL}(r+s)$ crystal $\mathcal{B}_\lambda^{(n)} \boxtimes B_\lambda^{(r+s)}$ to a $\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal. In doing so, $\mathcal{B}_\lambda^{(n)}$ stays the same, which means that the number of subcrystals isomorphic to $\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$ is equal to the multiplicity of $\mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal that results from branching $\mathcal{B}_\lambda^{(r+s)}$.

Now, we also note that

$$C \subset C_2 := \{X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) | Q'(X) \in \mathcal{B}_\mu^{(r)}, Q''(X) \in \mathcal{B}_\nu^{(s)}\}.$$

As was just proven in Proposition 9.3, $Q(X) \equiv Q'(X) \boxtimes Q''(X)$. Also, we have that $\mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) \equiv \mathrm{Mat}_{r\times n}(\mathbb{N}) \boxtimes \mathrm{Mat}_{s\times n}(\mathbb{N})$ by stacking the matrices. Therefore, we have that

$$C_2 \equiv \mathcal{B}_\mu^{(n)} \boxtimes \mathcal{B}_\nu^{(n)} \boxtimes \mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}.$$

Equivalently, we have the extra condition that $P'(X) \in \mathcal{B}_\mu^{(n)}$ and $P''(X) = \mathcal{B}_\nu^{(n)}$. By Proposition 9.3, $P(X) \equiv P'(X) \otimes P''(X)$. However, we also know that for $X \in C$, $P(X) \in \mathcal{B}_\lambda^{(n)}$. Therefore, the number of subcrystals isomorphic to $\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\mu^{(r)} \boxtimes \mathcal{B}_\nu^{(s)}$ equals the multiplicity of $\mathcal{B}_\lambda^{(n)}$ in $\mathcal{B}_\mu^{(n)} \otimes \mathcal{B}_\nu^{(n)}$. □

Example: Use the Littlewood-Richardson Rule to compute $c_{(2,1),(2,1)}^{(3,2,1)}$. In other words, we need to find the number of connected components of $\mathcal{B}_{(2,1)}^{(n)} \otimes \mathcal{B}_{(2,1)}^{(n)}$ that are isomorphic to $\mathcal{B}_{(3,2,1)}^{(n)}$. Here, n is chosen large enough so that all of the above are well-defined. For simplicity, we choose $n = 3$. $\mathcal{B}_{(2,1)}^{(3)}$ has 8 elements, namely $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$,

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$.

Instead of drqwing the crystal graph for all 64 elements of $\mathcal{B}_{(2,1)}^{(3)} \otimes \mathcal{B}_{(2,1)}^{(3)}$, we simply need to find the highest weight elements of this crystal that correspond to a higest weight element in the crystal $\mathcal{B}_{(3,2,1)}^{(3)}$, which has highest weight element $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$. To get the weight $(3,2,1)$, there are only six possibilities in $\mathcal{B}_{(2,1)}^{(3)} \otimes \mathcal{B}_{(2,1)}^{(3)}$ becase we need three $1s$, two $2s$, and one 3. The possibilities are $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$,

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$,

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Now, we count how many of these are highest weight. We do this by using the row reading and signature rule.

In the first case, the row readings give

$$2\ 1\ 1 \otimes 3\ 1\ 2.$$

We place a) correspinding to every 1 and a (correspinding to every 2 to check the 1−adjacency. This gives us

$$(\ )\ )\ \ )\ ($$

After cancellation, this results in))(, so we have $\varphi_1 = 2$ and $\varepsilon_1 = 1$. Hence, this element is not highest weight. We do this for all of these elements, using a) next to each 2 and a (next to each 3 to check the 2−adjacencies. The only elements that give $\varepsilon_1 = \varepsilon_2 = 0$ are $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$. To see this more clearly, we take the row readings in the first tensor above to get

$$3\ 1\ 2 \otimes 2\ 1\ 1.$$

To check the 1-adjacency, we place appropriate parenthesis to get

$$)\ (\ (\ )\ )\ ,$$

which reduces to). Hence, $\varphi_1 = 1$ and $\varepsilon_1 = 0$. To check the 2-adjacency, we place appropriate parenthesis to get

$$(\ )\ )\ ,$$

which also reduces to), giving $\varphi_2 = 1$ and $\varepsilon_2 = 0$. We get the same result when we exammine $2\ 1\ 3 \otimes 2\ 1\ 1$. Hence, $c^{(3,2,1)}_{(2,1),(2,1)} = 2$.