Math 846 Presentation notes Yu Sun Dec 11th

Section 9.2

$\boxed{\text{Intr of see-saw}}$

Ω is a vector space with actions G_1 & G_2, $G_1 G_2$ are groups & $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. H_2 commutes with G_1, H_1 commutes with G_2. G_1 & G_2 may not commute. Then Ω becomes either $G_1 \times H_2$-module or $G_2 \times H_1$-module. If we assume Ω is a correspondence for either of these actions.

$$\begin{tikzcd} G_1 \arrow[dr, dash] & G_2 \\ H_1 \arrow[u, hook] \arrow[ur, dash] & H_2 \arrow[u, hook] \end{tikzcd}$$

$\longleftarrow$ inclusion

correspondence.

Recall in Section 9.1

Thm 9.1: make $Mat_{r\times n}(\mathbb{N})$ into a $GL(n)\times GL(r)$ crystal.

Cor 9.2: As $GL(n)\times GL(r)$ crystals,

$$Mat_{r\times n}(\mathbb{N}) \cong \bigoplus_{\substack{\lambda \\ l(\lambda)\le \max\{r,n\}}} \left(\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\lambda^{(r)}\right)$$

Pf: Use the formulas in Thm 9.1 $\Rightarrow$ $X \longmapsto P(X) \boxtimes Q(X)$ is a morphism of $GL(n)\times GL(r)$ crystals, from $Mat_{r\times n}(\mathbb{N})$ to $\bigoplus_\lambda (\mathcal{B}_\lambda^{(n)} \boxtimes \mathcal{B}_\lambda^{(r)})$.

By 7.13 $(X \leftrightarrow (P(X), Q(X)))$, it's isomorphism.

$\square$

<u>To obtain crstal analog of see-saw:</u>

$$X = \begin{pmatrix} X'_{r\times n} \\ X''_{s\times n} \end{pmatrix} \in Mat_{(r+s)\times n}(\mathbb{N}) \quad \text{by stacking}$$

By Cor 9.2

$$X \qquad\qquad \begin{pmatrix} X' \\ X'' \end{pmatrix} \qquad X' \boxtimes X'' \in Mat_{r\times n} \boxtimes Mat_{s\times n}$$

$GL(n)\times GL(r+s)$ crystal structure

$GL(n)\times GL(r)\times GL(n)\times GL(s)$ structure

$\downarrow$ regarded as

commuting $GL(n)\times GL(n)\times GL(r)\times GL(s)$ structure

Then we have the see-saw

$$\begin{tikzcd} GL(n)\times GL(n) \arrow[d, leftrightarrow] \arrow[drr, dash] & & GL(r+s) \arrow[d, leftrightarrow] \arrow[dll, dash] \\ GL(n) & & GL(r)\times GL(s). \end{tikzcd} \qquad (9.2)$$

But why $GL(n)\times GL(n)$ & $GL(r)\times GL(s)$ are commutative, $GL(n)$ & $GL(r+s)$

but $GL(n)\times GL(n)$ & $GL(r+s)$ are not, see Appendix.
(For having better understanding of see-saw in representation point of view, see book Appendix A.4. B.1, B.2)

Next, we need to check those two structures are <u>comptable</u> with tensor product operation, on $GL(n)$ crystals and Levi branching from $GL(r+s)$ to $GL(r)\times GL(s)$

<u>Prop 9.3</u>: Above $\begin{pmatrix} X' \\ X'' \end{pmatrix} = X$,

$X \xleftrightarrow{RSK} (P, Q)$

$X' \xleftrightarrow{RSK} (P', Q')$

$X'' \xleftrightarrow{RSK} (P'', Q'')$

then $P \equiv P' \otimes P''$, $Q \equiv Q' \boxtimes Q''$

<u>Pf: For P.</u>

R_j: row tableau containing $x_{j,i}$ opies of $\boxed{i}$
$1\le j\le r+s$, $1\le i\le n$

R_j' : $\quad$: $\quad$:
$1\le j\le r$

R_j'' : $\quad$: $\quad$:
$r+1\le j\le r+s$

(Thm 8.6. $\mathbb{B}^{\otimes k}$ decomposes into disjoint union of crystals, isomorphic to B_λ, λ is a part$\overset{\text{ition}}{}$ of k, $l(\lambda)\le n$, $x\in \mathbb{B}^{\otimes k}$,
(i) $x \equiv P(x)$)

$$\left.\begin{aligned} \Rightarrow P &= R_1 \otimes \cdots \otimes R_{r+s} \\ P' &= R_1 \otimes \cdots \otimes R_r \\ P'' &= R_{r+1} \otimes \cdots \otimes R_{r+s} \end{aligned}\right\} \Rightarrow P \equiv P' \otimes P'' \text{ from RSK of constructing } P.$$

i.e. $X = \left(\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right)_{\substack{(r+s)\times n \\ 4\times 3 \\ (2+2)}}$

$R_1 = \boxed{1\,|\,1\,|\,2} = \boxed{1}\otimes\boxed{1}\otimes\boxed{2} = R_1'$

$R_2 = \boxed{1\,|\,3} = \boxed{1}\otimes\boxed{3} = R_2'$

$R_3 = \boxed{2\,|\,3} = \boxed{2}\otimes\boxed{3} = R_3''$

$R_4 = \boxed{1\,|\,3\,|\,3} = \boxed{1}\otimes\boxed{3}\otimes\boxed{3} = R_4''$

$\widetilde{R}_1 = \boxed{1\,|\,1\,|\,2\,|\,4} = \boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4}$

$\widetilde{R}_2 = \boxed{1\,|\,3} = \boxed{1}\otimes\boxed{3}$

$\widetilde{R}_3 = \boxed{2\,|\,3\,|\,4\,|\,4} = \boxed{2}\otimes\boxed{3}\otimes\boxed{4}\otimes\boxed{4}$

$\widetilde{R}_1' = \boxed{1\,|\,1\,|\,2}$ $\qquad$ $\widetilde{R}_1'' = \boxed{4}$

$\widetilde{R}_2' = \boxed{1}$ $\qquad$ $\widetilde{R}_2'' = \boxed{3}$

$\widetilde{R}_3' = \boxed{2}$ $\qquad$ $\widetilde{R}_3'' = \boxed{3\,|\,4\,|\,4} = \boxed{3}\otimes\boxed{4}\otimes\boxed{4}$

For Q

$\widetilde{R}_j$: denotes row tableau containing $x_{i,j}$ copies of $\boxed{i}$
$1 \le j \le n$ $\qquad 1 \le i \le r+s$

$\widetilde{R}'_j$: " " $\boxed{i}$
$1 \le j \le n$ $\qquad 1 \le i \le r$

$\widetilde{R}''_j$: " " $\boxed{i}$
$1 \le j \le n$ $\qquad r+1 \le i \le r+s$

On branching to $GL(r) \times GL(s)$ crystals

$$\widetilde{R}_j = \widetilde{R}'_j \otimes \widetilde{R}''_j \underset{GL(r)\times GL(s)}{\equiv} \widetilde{R}'_j \boxtimes \widetilde{R}''_j \quad (GL(r) \times GL(s) \equiv \text{coarser than } GL(r+s))$$

$$\Rightarrow Q \underset{GL(r)\times GL(s)}{\equiv} (\widetilde{R}'_1 \boxtimes \widetilde{R}''_1) \otimes \cdots \otimes (\widetilde{R}'_n \boxtimes \widetilde{R}''_n)$$

$\Downarrow$

$$Q \equiv (\widetilde{R}'_1 \otimes \cdots \otimes \widetilde{R}'_n) \boxtimes (\widetilde{R}''_1 \otimes \cdots \otimes \widetilde{R}''_n) = Q' \boxtimes Q''$$

$\square$

Appendix.

Prop 1. $GL(n) \times GL(n)$ & $GL(r+s)$ are not commatative.

Pf: $U = \begin{pmatrix} U' \\ U'' \end{pmatrix} \in Mat_{(r+s)\times n}(\mathbb{C})$, $U' \in Mat_{r\times n}(\mathbb{C})$, $U'' \in Mat_{s\times n}(\mathbb{C})$

g: left multiplication: $\begin{pmatrix} U' \\ U'' \end{pmatrix} \longrightarrow \begin{pmatrix} U'\Upsilon' \\ U''\Upsilon'' \end{pmatrix}$, $\Upsilon', \Upsilon'' \in Mat_{n\times n}(\mathbb{C})$

h: right multiplication: $\begin{pmatrix} U' \\ U'' \end{pmatrix} \longrightarrow A \begin{pmatrix} U' \\ U'' \end{pmatrix}$, $A \in Mat_{(r+s)\times(r+s)}(\mathbb{C})$

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

$$\Rightarrow h(gu) = h\begin{pmatrix} U'\Upsilon' \\ U''\Upsilon'' \end{pmatrix} = \begin{pmatrix} A_1U'\Upsilon' + A_2U''\Upsilon' \\ A_3U'\Upsilon' + A_4U''\Upsilon'' \end{pmatrix}$$

$$g(hu) = g\begin{pmatrix} A_1U' + A_2U'' \\ A_3U' + A_4U'' \end{pmatrix} = \begin{pmatrix} A_1U'\Upsilon' + A_2U''\Upsilon' \\ A_3U'\Upsilon'' + A_4U''\Upsilon'' \end{pmatrix}$$

$h(gu) \neq g(hu)$ in general unlease $\Upsilon' = \Upsilon''$

$\square$

Prop 2.: $GL(n) \times GL(n)$ & $GL(r) \times GL(s)$ are commatative.
$GL(n)$ & $GL(r+s)$

Pf: Same as above. Leave it to you as an exercise.

$\square$