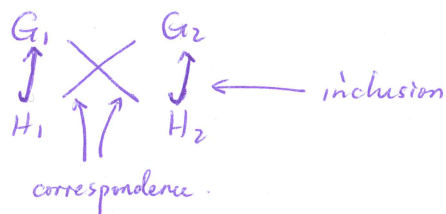


Section 9.2

Intr of see-saw

Ω is a vector space with actions G_1 & G_2 , G_1, G_2 are groups & $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. H_2 commutes with G_1 , H_1 commutes with G_2 . G_1 & G_2 may not commute. Then Ω becomes either $G_1 \times H_2$ -module or $G_2 \times H_1$ -module. If we assume Ω is a correspondence for either of these actions.



Recall in Section 9.1

Thm 9.1: make $\text{Mat}_{r \times n}(\mathbb{N})$ into a $GL(n) \times GL(r)$ crystal.

Cor 9.2: As $GL(n) \times GL(r)$ crystals,

$$\text{Mat}_{r \times n}(\mathbb{N}) \cong \bigoplus_{\lambda \in \max\{r, n\}} (B_\lambda^{(n)} \boxtimes B_\lambda^{(r)})$$

Pf: Use the formulas in Thm 9.1 $\Rightarrow X \mapsto P(X) \boxtimes Q(X)$ is a morphism of $GL(n) \times GL(r)$ crystals, from $\text{Mat}_{r \times n}(\mathbb{N})$ to $\bigoplus_{\lambda} (B_\lambda^{(n)} \boxtimes B_\lambda^{(r)})$.

By 7.13 ($X \leftrightarrow (P(X), Q(X))$), it's isomorphism. □

To obtain crystal analog of see-saw:

$$X = \begin{pmatrix} X'_{r \times n} \\ X''_{s \times n} \end{pmatrix} \in \text{Mat}_{(r+s) \times n}(\mathbb{N}) \quad \text{by stacking}$$

By Cor 9.2

X

$GL(n) \times GL(r+s)$ crystal structure

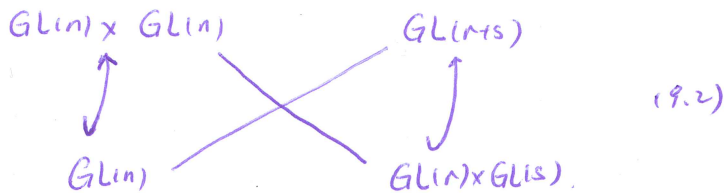
$$\begin{pmatrix} X' \\ X'' \end{pmatrix}$$

$$X' \boxtimes X'' \in \text{Mat}_{r \times n} \boxtimes \text{Mat}_{s \times n}$$

$GL(n) \times GL(r) \times GL(n) \times GL(s)$ structure
↓ regarded as

commuting $GL(n) \times GL(n) \times GL(r) \times GL(s)$ structure

Then we have the see-saw



But why $\text{GL}(n) \times \text{GL}(n)$ & $\text{GL}(n) \times \text{GL}(s)$ are commutative,
 $\text{GL}(n)$ & $\text{GL}(n+s)$

but $\text{GL}(n) \times \text{GL}(n)$ & $\text{GL}(n+s)$ are not, see Appendix.
 (For having better understanding of see-saw in representation point of view, see book Appendix A.4. B.1, B.2)

Next, we need to check those two structures are compatible with tensor product operation, on $\text{GL}(n)$ crystals and Levi branching from $\text{GL}(n+s)$ to $\text{GL}(n) \times \text{GL}(s)$

Prop 9.3: Above $\begin{pmatrix} X' \\ X'' \end{pmatrix} = X$,

$$X \xrightarrow{\text{RSK}} (P, Q)$$

$$X' \xleftarrow{\text{RSK}} (P', Q')$$

$$X'' \xleftarrow{\text{RSK}} (P'', Q'')$$

$$\text{then } P \equiv P' \otimes P'', \quad Q \equiv Q' \boxtimes Q''$$

Pf: For P ,

R_j : row tableau containing $x_{j,i}$ copies of $\square_{i \leq i \leq n}$

R'_j :
 $i \in j \in r$

R''_j :
 $r \in j \in s$

(Thm 8.6. $B^{\otimes k}$ decomposes into disjoint union of crystals, isomorphic to B_λ , λ is a part of k , $\lambda \in \mathbb{N}^n$, $x \in B^{\otimes k}$,

$$(i) \ x \equiv P(x)$$

$$\Rightarrow P = R_1 \otimes \dots \otimes R_{n+s}$$

$$P' = R_1 \otimes \dots \otimes R_r$$

$$P'' = R_{r+1} \otimes \dots \otimes R_{n+s}$$

$\Rightarrow P \equiv P' \otimes P''$ from RSK of constructing P .

i.e. $X = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{matrix} (r+s) \times n \\ 4 \times 3 \\ (2+2) \end{matrix}$

$$R_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} = R_1'$$

$$R_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} = R_2'$$

$$R_3 = \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} = R_3''$$

$$R_4 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} = R_4''$$

$$\tilde{R}_1 = \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix}$$

$$\tilde{R}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix}$$

$$\tilde{R}_3 = \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix}$$

$$\tilde{R}_1' = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \quad \tilde{R}_1'' = \begin{bmatrix} 2 \end{bmatrix}$$

$$\tilde{R}_2' = \begin{bmatrix} 1 \end{bmatrix} \quad \tilde{R}_2'' = \begin{bmatrix} 2 \end{bmatrix}$$

$$\tilde{R}_3' = \begin{bmatrix} 2 \end{bmatrix} \quad \tilde{R}_3'' = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \end{bmatrix}$$

For \mathcal{Q}

\tilde{R}_j : denotes row tableau containing $x_{i,j}$ copies of \square
 $1 \leq j \leq n$ $1 \leq i \leq m$

\tilde{R}_j : \square
 $1 \leq j \leq n$ $1 \leq i \leq r$

\tilde{R}_j'' : \square
 $1 \leq j \leq n$ $m+1 \leq i \leq m+s$

On branching to $GL(m) \times GL(s)$ crystals

$$\tilde{R}_j = \tilde{R}_j' \otimes \tilde{R}_j'' \equiv_{GL(m) \times GL(s)} \tilde{R}_j' \boxtimes \tilde{R}_j'' \quad (GL(m) \times GL(s) \equiv \text{coarser than } GL(m+s))$$

$$\Rightarrow \mathcal{Q} \equiv_{GL(m) \times GL(s)} (\tilde{R}_1' \boxtimes \tilde{R}_1'') \otimes \dots \otimes (\tilde{R}_n' \boxtimes \tilde{R}_n'')$$

\Downarrow

$$\mathcal{Q} \equiv (\tilde{R}_1' \otimes \dots \otimes \tilde{R}_n') \boxtimes (\tilde{R}_1'' \otimes \dots \otimes \tilde{R}_n'') = \mathcal{Q}' \boxtimes \mathcal{Q}''$$

□

Appendix.

Prop 1. $GL(m) \times GL(n)$ & $GL(r+s)$ are not commutative.

Pf: $u = \begin{pmatrix} u' \\ u'' \end{pmatrix} \in \text{Mat}_{(r+s) \times n}(\mathbb{C})$, $u' \in \text{Mat}_{r \times n}(\mathbb{C})$, $u'' \in \text{Mat}_{s \times n}(\mathbb{C})$

g: left multiplication: $\begin{pmatrix} u' \\ u'' \end{pmatrix} \rightarrow \begin{pmatrix} u' \gamma' \\ u'' \gamma'' \end{pmatrix}$, $\gamma', \gamma'' \in \text{Mat}_{n \times n}(\mathbb{C})$

h: right multiplication: $\begin{pmatrix} u' \\ u'' \end{pmatrix} \rightarrow A \begin{pmatrix} u' \\ u'' \end{pmatrix}$, $A \in \text{Mat}_{(r+s) \times (r+s)}(\mathbb{C})$

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

$$\Rightarrow h(gu) = h \begin{pmatrix} u' \gamma' \\ u'' \gamma'' \end{pmatrix} = \begin{pmatrix} A_1 u' \gamma' + A_2 u'' \gamma'' \\ A_3 u' \gamma' + A_4 u'' \gamma'' \end{pmatrix}$$

$$g(hu) = g \begin{pmatrix} A_1 u' + A_2 u'' \\ A_3 u' + A_4 u'' \end{pmatrix} = \begin{pmatrix} A_1 u' \gamma' + A_2 u'' \gamma'' \\ A_3 u' \gamma'' + A_4 u'' \gamma'' \end{pmatrix}$$

$h(gu) \neq g(hu)$ in general unless $\gamma' = \gamma''$

□

Prop 2.: $GL(m) \times GL(n)$ & $GL(m) \times GL(s)$ are commutative.
 $GL(n)$ & $GL(r+s)$

Pf: Same as above. Leave it to you as an exercise.

□