$GL(n) \times GL(r)$ bicrystal

Ian Seong, Feng Sheng

Introduction

In Chapter 7, we showed that there is a bijection between $\mathrm{Mat}_{r\times n}(\mathbb{N})$ and the set of pairs (P, Q). We are going to realize the crystals formed by the elements of $\mathrm{Mat}_{r\times n}(\mathbb{N})$, which turns out to be $GL(n) \times GL(r)$, due to the crystal structures of P and Q.

Construction

In this section, we will show how to construct functions like $\varphi_i, \varepsilon_i, f_i, e_i$, using both general construction and an example.

We first implement the crystal structure of $GL(n)$ on $\mathrm{Mat}_{r\times n}(\mathbb{N})$.

Let X be an $r \times n$ matrix. In this section, we will use an example

$$X_0 = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

We define an intermediate variable $\psi_i(X)$ as the following:

$$\psi_i(X) = (\psi_{i,1}(X), \cdots, \psi_{i,r}(X)), \text{where } \psi_{i,j}(X) = \sum_{k\leq j} x_{k,i} - \sum_{k<j} x_{k,i+1}$$

Using the example,

$$\psi_{1,1}(X_0) = \sum_{k\leq 1} x_{k,1} - \sum_{k<1} x_{k,2} = 1$$

$$\psi_{1,2}(X_0) = \sum_{k\leq 2} x_{k,1} - \sum_{k<2} x_{k,2} = (1+0) - (2) = -1$$

$$\psi_{1,3}(X_0) = \sum_{k\leq 3} x_{k,1} - \sum_{k<3} x_{k,2} = (1+0+3) - (2-1) = 1$$

Therefore, $\psi_1(X) = (1, -1, 1)$.

Using the same method, we can obtain $\psi_2(X) = (2, 3, 1)$ and $\psi_3(X) = (0, 1, 0)$.

Now, we will define $\varphi_i(X)$ as the greatest coordinate in $\psi_i(X)$.

Hence, in the example, $\varphi_1(X_0) = 1$, $\varphi_2(X_0) = 3$, and $\varphi_3(X_0) = 1$.

Now, if $\varphi_i(X) = 0$, then $f_i(X) = \phi$ (where ϕ is the special symbol). If $\varphi_i(X) > 0$, we look at the first coordinate that gives $\varphi_i(X)$ (call this place j), and this is where we, to obtain $f_i(X)$, subtract from the j-th row of the matrix the i-th simple root, $(0, \cdots, 0, 1, -1, 0, \cdots, 0)$, where 1 is at the i-th coordinate.

In our example, if $i = 1$, we can see that the first coordinate gives us $\varphi_1(X_0)$, so we subtract $(1, -1, 0, 0)$ from the first row, giving us

$$f_1(X_0) = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, if $i = 2$, the second coordinate gives us $\varphi_2(X_0)$, so

$$f_2(X_0) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

Lastly,

$$f_3(X_0) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

We can define $e_i(X)$ in a similar fashion, just the exact opposite.

Introduce the intermediate variable $\delta_i(X)$, which is defined by

$$\delta_i(X) = (\delta_{i,1}(X), \delta_{i,2}(X), \cdots, \delta_{i,r}(X)), \text{where } \delta_{i,j}(X) = \sum_{k \geq j} x_{k,i+1} - \sum_{k > j} x_{k,i}.$$

Looking at our example, we obtain

$$\delta_1(X_0) = (0, -2, 0)$$
$$\delta_2(X_0) = (1, 2, 0)$$
$$\delta_3(X_0) = (1, 2, 1).$$

Now, $\varepsilon_i(X)$ is defined by the maximal coordinate in $\delta_i(X)$. Hence, $\varepsilon_1(X_0) = 0$, $\varepsilon_2(X_0) = 2$, and $\varepsilon_3(X_0) = 2$.

To define $e_i(X)$, if $\varepsilon_i(X) = 0$, then $e_i(X) = \phi$. Otherwise, we look at the LAST coordinate that gives $\varepsilon_i(X)$ (call this place j) and add the i-th simple root to row j of the matrix X.

In our example, $\varepsilon_1(X_0) = 0$, so $e_1(X_0) = \phi$.

The second coordinate of $\delta_2(X_0)$ gives us $\varepsilon_2(X_0) = 2$, so we add the second simple root $(0, 1, -1, 0)$ to the second row, giving us

$$e_2(X_0) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$e_3(X_0) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

To construct $GL(r)$, we can transpose the matrix X, and establish the same construction as above to X^T. Call the operators f_i' and e_i'.

Theorem

Theorem (9.1)**.** *The constructions above realize* $Mat_{r\times n}$ *as a* $GL(n)\times GL(r)$ *crystal. The operators* f_i*,* e_i*,* f_i' *and* e_i' *satisfy the following:*

$$\begin{aligned} P(f_i(X)) &= f_i(P(X)) & P(e_i(X)) &= e_i(P(X)) \\ Q(f_i(X)) &= Q(X) & Q(e_i(X)) &= Q(X) \\ Q(f_i'(X)) &= f_i'(Q(X)) & Q(e_i'(X)) &= e_i'(Q(X)) \\ P(f_i'(X)) &= P(X) & P(e_i'(X)) &= P(X) \end{aligned}$$

Proof. The last two rows of the identities follow from the first two rows, due to transpose (see Theorem 7.14 in the appendix).

Therefore, it suffices to just prove for the first two rows.

We have established the bijection between $\mathrm{Mat}_{r\times n}(\mathbb{N})$ and (P, Q) in Theorem 7.13. In fact, we can establish the crystal structure on $\mathrm{Mat}_{r\times n}(\mathbb{N})$ via exporting the structure from $P(X)$ and having f_i and e_i act trivially on $Q(X)$.

We will show that this establishment has the weight function, φ_i, ε_i, f_i, and e_i agree with the construction we have in the previous section.

Suppose R_j is the row tableau with x_{jk} copies of the box $\boxed{k}$, where x_{jk} is the j-th row k-th column entry of X.

Going back to the example of $X_0 = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}$,

$$R_1 = \boxed{1}\boxed{2}\boxed{2}\boxed{4}$$

$$R_2 = \boxed{2}\boxed{3}\boxed{3}\boxed{4}$$

$$R_3 = \boxed{1}\boxed{1}\boxed{1}\boxed{4}$$

By Theorems 8.5 and 8.6 in the appendix, we can see that $R_1 \otimes \cdots \otimes R_r$ is plactically equivalent to $P(X)$. Hence, by Lemma 2.33,

$$\varphi_i(P(X)) = \varphi_i(R_1 \otimes \cdots \otimes R_r) = \max_{j=1}^{r} \left(\sum_{k\le j} x_{k,i} - \sum_{k<j} x_{k,i+1} \right) = \max_{j=1}^{r} \{\psi_{i,j}(X)\}.$$

In our example,

$$P(X_0) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3 & 3 & 4 & 4 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$$

Using the signature rule, we can see that $\varphi_1(P(X_0)) = 1, \varphi_2(P(X_0)) = 3$, and $\varphi_3(P(X_0)) = 1$, which aligns with our previous example.

Similarly,

$$\varepsilon_i(P(X)) = \varepsilon_i(R_1 \otimes \cdots \otimes R_r) = \max_{j=1}^{r} \left(\sum_{k \geq j} x_{k,i+1} - \sum_{k > j} x_{k,i} \right) = \max_{j=1}^{r} \{\delta_{i,j}(X)\}.$$

Also from Lemma 2.33, we can see that

$$f_i(R_1 \otimes R_2 \otimes \cdots \otimes R_r) = R_1 \otimes \cdots \otimes f_i(R_j) \otimes \cdots \otimes R_r$$

and

$$e_i(R_1 \otimes R_2 \otimes \cdots \otimes R_r) = R_1 \otimes \cdots \otimes e_i(R_j) \otimes \cdots \otimes R_r.$$

By Theorem 8.6, $f_i(X)$ is plactically equivalent to $P(f_i(X))$, and since X is plactically equivalent to $P(X)$, $P(f_i(X)) \equiv f_i(X) \equiv f_i(P(X))$.

Similarly, $P(e_i(X)) \equiv e_i(P(X))$.

Finally, we need to show that Q is preserved by f_i and e_i.

By Theorem 8.7, we can see that since X, $f_i(X)$, and $e_i(X)$ all lie in the same connected subcrystal, $\overline{Q}(X) = \overline{Q}(f_i(X)) = \overline{Q}(e_i(X))$, where $\overline{Q}$ is a recording tableau, and hence, $\overline{Q}$ is a standard (not semistandard) tableau. We will obtain a semistandard tableau Q from this.

Let s_i be the row sum of i-th row in X. From $\overline{Q}(x)$, we can obtain a semistandard tableau $Q(X)$ by replacing the first s_1 numbers by 1, the next s_2 numbers by 2, and so on. Therefore,

$$Q(X) = Q(f_i(X)) = Q(e_i(X)).$$

In our example, consider $\overline{Q}(f_1(X_0))$, which is

$\overline{Q}(X) =$

1	2	3	4	6	7	8	12
5	10	11					
9							

Since the sum of the entries in the first row is 4 (and same for the second and third rows), the semistandard tableau Q is

$\overline{Q}(X_0) =$

1	1	1	1	2	2	2	3
2	3	3					
3							

Hence, $\mathrm{Mat}_{r \times n}$ can be realized as a $GL(n) \times GL(r)$ crystal with our desired properties.

$\square$

Appendix

Preliminary Lemmas and Theorems

Lemma (2.33)**.** *Let $x_1 \otimes \cdots \otimes x_k \in \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_k$ for finite type crystals $\mathcal{B}_1, \cdots, \mathcal{B}_k$. Then,*

$$\varphi_i(x_1 \otimes \cdots \otimes x_k) = \max_{j=1}^{k} \left(\sum_{h=1}^{j} \varphi_i(x_h) - \sum_{h=1}^{j-1} \varepsilon_i(x_h) \right),$$

and if j is the first place where the maximum is attained, then

$$f_i(x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes f_i(x_j) \otimes \cdots \otimes x_k.$$

Similarly, for $x_k \otimes \cdots \otimes x_1 \in \mathcal{B}k \otimes \cdots \otimes \mathcal{B}_1$,

$$\varepsilon_i(x_k \otimes \cdots \otimes x_1) = \max_{j=1}^{k} \left(\sum_{h=1}^{j} \varepsilon_i(x_h) - \sum_{h=1}^{j-1} \varphi_i(x_h) \right),$$

and if j is the last place where the maximum is attained, then

$$e_i(x_k \otimes \cdots \otimes x_1) = x_k \otimes \cdots \otimes e_i(x_j) \otimes \cdots \otimes x_1.$$

Theorem (7.13)**.** *The map from $M_{r\times n}(\mathbb{N})$ to pairs (P, Q) of semistandard tableaux with the same shape λ, where P is in the alphabet $[n]$ and Q is in the alphabet $[r]$, is a bijection.*

Theorem (7.14)**.** *If $X \in M_{r\times n}(\mathbb{N})$ maps to (P, Q), then the transpose X^T maps to (Q, P).*

Theorem (8.5)**.** *If $T \in \mathcal{B}_{(k)}$ is a row tableau, then $T \otimes \boxed{j} \equiv T \leftarrow j$, where $\equiv$ refers to plactic equivalence.*

Theorem (8.6)**.** *The crystal $\mathbb{B}^{\otimes k}$ decomposes into a disjoint union of crystals, each isomorphic to λ, where λ is a partition of k of length at most n. Suppose $x \in \mathbb{B}^{\otimes k}$.*

1. *$x \equiv P(x)$*
2. *If λ is a shape of $P(x)$ and $Q(x)$, then x lies in a subcrystal isomorphic to $\mathcal{B}_\lambda$.*
3. *If $P(x) = P(y)$, then x and y are plactically equivalent.*

Theorem (8.7)**.** *Suppose $x, y \in \mathbb{B}^{\otimes k}$. Then, $Q(x) = Q(y)$ if and only if x and y lie in the same connected subcrystal.*