LEVI BRANCHING OF $\mathcal{B}_\lambda$ FROM $\mathrm{GL}(n)$ TO $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$

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Recall that [1] Section 2.8 described the Levi branching: suppose $\mathcal{B}$ is a crystal for the root system Φ and J is a subset of the index set I for Φ, then deleting $f_i, e_i, \varphi_i, \varepsilon_i$ from B gives a Φ_J crystal B_J.

Note that $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ is naturally a subgroup of $\mathrm{GL}(n)$, and their weight lattices satisfy

$$\Lambda_{\mathrm{GL}(n)} = \mathbb{Z}^n \cong \mathbb{Z}^r \times \mathbb{Z}^{n-r} = \Lambda_{\mathrm{GL}(r)} \times \Lambda_{\mathrm{GL}(n-r)}.$$

Let $I = \{1, 2, \cdots, n-1\}$ be the index set for $\mathrm{GL}(n)$, then $I \setminus \{r\}$ is the index set for $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$, where the simple roots $\alpha_1, \cdots, \alpha_{r-1}$ are identified as simple roots for $\mathrm{GL}(r)$, and $\alpha_{r+1}, \cdots, \alpha_{n-1}$ are identified as simple roots for $\mathrm{GL}(n-r)$. On top of this, let $\mathcal{C}$ and $\mathcal{D}$ be connected Stembridge crystals for $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ respectively, then $\mathcal{C} \boxtimes \mathcal{D}$ is a connected Stembridge crystal for $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$. Here as a set

$$\mathcal{C} \boxtimes \mathcal{D} = \{x \boxtimes y \mid x \in \mathcal{C}, y \in \mathcal{D}\}$$

is the Cartesian product of $\mathcal{C}$ and $\mathcal{D}$,

$$\mathrm{wt}(x \boxtimes y) = (\mathrm{wt}(x), \mathrm{wt}(y)),$$

$$f_i(x \boxtimes y) = \begin{cases} f_i(x) \boxtimes y, & \text{if } i < r, \\ x \boxtimes f_i(y), & \text{if } i > r, \end{cases}$$

$$\varphi_i(x \boxtimes y) = \begin{cases} \varphi_i(x), & \text{if } i < r, \\ \varphi_i(y), & \text{if } i > r, \end{cases}$$

and the definitions for e_i and ε_i are similar.

The above shows how to construct connected Stembridge $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ crystals from connected Stembridge $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ crystals. In fact, it turns out that this construction exhausts all connected Stembridge $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ crystals, up to isomorphism.

Lemma 1. *Every connected Stembridge $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ crystals are of the form $\mathcal{C} \boxtimes \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are connected Stembridge $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ crystals respectively.*

Proof. Given any connected Stembridge $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ crystal $\mathcal{E}$, consider its highest weight $\mu \in \Lambda_{\mathrm{GL}(r)} \times \Lambda_{\mathrm{GL}(n-r)}$. Write $\mu = (\mu', \mu'')$ where $\mu' \in \Lambda_{\mathrm{GL}(r)}$ and $\mu'' \in \Lambda_{\mathrm{GL}(n-r)}$. Note that μ must be a dominant weight for $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ since Stembridge crystals are seminormal, so μ' and μ'' are also dominant weights for $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ respectively. Hence there are connected Stembridge $\mathrm{GL}(r)$ and $\mathrm{GL}(n-r)$ crystals $\mathcal{C}$ and $\mathcal{D}$ with highest weights μ' and μ'' respectively. Consider $\mathcal{C} \boxtimes \mathcal{D}$ as constructed above, it is isomorphic to $\mathcal{E}$ by [1] Theorem 4.13, since $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{E}$ have the same highest weight. ☐

Next we consider the Levi branching of crystals of tableaux. Again we have $\Lambda_{\mathrm{GL}(n)} = \Lambda_{\mathrm{GL}(r)} \times \Lambda_{\mathrm{GL}(n-r)}$.

Theorem 2. *Suppose λ is a partition, $|\lambda| = k$, and $l(\lambda) \leq n$. Branching $\mathcal{B}_\lambda$ to* $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ *gives*

$$\mathcal{B}_\lambda \cong \bigoplus_{\substack{l(\mu)\leq r\\ \mathrm{YD}(\mu)\subseteq\mathrm{YD}(\lambda)}} \mathcal{B}_\mu \boxtimes \mathcal{B}_{\lambda/\mu} \cong \bigoplus_{\substack{|\mu|+|\nu|=k\\ l(\mu)\leq r\\ l(\nu)\leq n-r}} (\mathcal{B}_\mu \boxtimes \mathcal{B}_\nu)^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. Given any tableau $T \in \mathcal{B}_\lambda$, T is semistandard, so all boxes with values $\leq r$ form a tableu of shape μ, the remaining boxes form a skew tableau of skew shape λ/μ, and both of them are still semistandard. This gives a bijection of sets

$$\mathcal{B}_\lambda \cong \bigoplus_{\substack{l(\mu)\leq r\\ \mathrm{YD}(\mu)\subseteq\mathrm{YD}(\lambda)}} \mathcal{B}_\mu \boxtimes \mathcal{B}_{\lambda/\mu}.$$

Since the construction of $\boxtimes$ preserves $\mathrm{wt}, e_i, f_i, \varepsilon_i, \varphi_i$ as expected, this is in fact an isomorphism of $\mathcal{B}_\lambda$ to $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ crystals.
By the property

$$\mathcal{B}_{\lambda/\mu} \cong \bigoplus_\nu \mathcal{B}_\nu^{\oplus c^\lambda_{\mu\nu}},$$

we have

$$\bigoplus_{\substack{l(\mu)\leq r\\ \mathrm{YD}(\mu)\subseteq\mathrm{YD}(\lambda)}} \mathcal{B}_\mu \boxtimes \mathcal{B}_{\lambda/\mu} \cong \bigoplus_{\substack{|\mu|+|\nu|=k\\ l(\mu)\leq r\\ l(\nu)\leq n-r}} (\mathcal{B}_\mu \boxtimes \mathcal{B}_\nu)^{\oplus c^\lambda_{\mu\nu}}.$$

□

Remark. The result of this theorem gives

$$s_\lambda(t_1, \cdots, t_n) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(t_1, \cdots, t_r) s_\nu(t_{r+1}, \cdots, t_n).$$

Note that s_λ is symmetric, so $c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu}$.
Futhermore, this result can be used to prove the identity

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

which is equivalent to

$$\langle s_\mu s_\nu, s_\lambda \rangle = c^\lambda_{\mu\nu}.$$

Here $\langle\ ,\ \rangle$ is the inner product defined on the ring of symmetric functions such that the Schur functions s_λ form an orthonormal basis for this ring, i.e. we define $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ and extend it to all of this ring.
Note that the property

$$\mathcal{B}_{\lambda/\mu} \cong \bigoplus_\nu \mathcal{B}_\nu^{\oplus c^\lambda_{\mu\nu}},$$

gives the identity

$$s_{\lambda/\mu} = \sum_\lambda c^\lambda_{\mu\nu} s_\nu,$$

which is equivalent to

$$\langle s_{\lambda/\mu}, s_\nu \rangle = c^\lambda_{\mu\nu}.$$

This is compatible with the common definition

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\mu s_\nu, s_\lambda \rangle$$

of $s_{\lambda/\mu}$ from the perspective of symmetric functions, for example as in [2].

References

[1] D.Bump, A.Schilling. *Crystal Bases: representations and combinatorics*. World Scientific; New Jersey, 2016.

[2] I.Macdonald. *Symmetric Functions and Hall Polynomials, Second Edition*. Oxford University Press; New York, 2015.