§8.4 CRYSTAL OF SKEW TABLEAUX

- Intro & Definitions

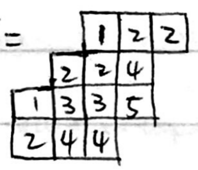
- Skew shape: (Recall we did this in Chp 7 for horizontal & vertical strip) is an ordered pair of partitions (λ, μ) s.t. $YD(\lambda) \supseteq YD(\mu)$. Denote λ/μ
- Young diagram $YD(\lambda/\mu)$: Set theoretic difference $YD(\lambda) \setminus YD(\mu)$.
- Skew Tableaux with shape λ/μ: assignment of values from $[n]$ to the boxes in $YD(\lambda/\mu)$.
- Semistandard: (Same as How we define for Young tableaux). rows are weakly increasing and the columns are strictly increasing.

NOTE: (How this is related to the usual ss tableaux). The usual ss tab is the special case $\mu = \emptyset$.

- Reading word, word(T) of a tableau T is the set of entries read by rows, from bottom to top ↑ and read each row left to right ⟶.
- Row reading $RR(T) \in B^{\otimes k}$, $k = |\lambda| - |\mu|$, obtained by tensoring elements in reading word.
(This is similar constructions we had in Chp 3 for tableau).
- With n fixed. $B_{\lambda/\mu}$:= set of ss. skew tableaux of shape λ/μ.

Example: Skew shape $\lambda/\mu = (3,4,4,3)/(2,1)$, so $YD(\lambda/\mu) =$ (Erase).

consider tableau T =



word(T) = 2441335224122 ⟶

$RR(T) = \boxed{2} \otimes \boxed{4} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{5} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2}$.

** OK, I just realized that this is too long to write on board. So insted I'm gonna do

$\lambda/\mu = (3,3,2,1)/(2,1)$

with T =

So word(T) = 213221

$RR(T) =$

$\boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1}$.

Theorem 8.8 The subset $RR(B_{\lambda/\mu})$ is a subcrystal of $B^{\otimes k}$.

Proof check compatibility with e_i f_i ε_i φ_i

Recall earlier in thm 3.2. we showed that $RR(B_\lambda)$, of ss. tabs of shape λ, is a connected component of $B^{\otimes k}$, (so stronger). so these two proofs are similar. □

- The IMPOTANCE here is. Trough embedding $B_{\lambda/\mu}$ in $B^{\otimes k}$ via RR, we equip it with crystal structure. So we defined Crystals of skew tableaux.
- So here $B_{\lambda/\mu}$ might not be as nice as B_λ. since the latter are embeddes into a connected component of $B^{\otimes k}$. Recall that (Cor 6.2). any conn. compont of $B^{\otimes k}$ is isomorphic to B_ν for some $\nu \vdash k$. In our situation. we get $B_{\lambda/\mu}$. can be decompose into direct sum of crystals iso. to B_ν (philosophy: skew $\rightarrow$ usual).
- Question: How to describe this decomposition?

Def: λ/μ skew shape. ν another partition. the Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$ is defined to be the number of skew tableaux of shape λ/μ with weight ν whose reading words are Yamanouchi words.

$\hookrightarrow$ we defined this on Monday: the word $u_1 \cdots u_k$ is Yamanouchi word if $\forall i$, each final segment $\{u_j, \ldots, u_k\}$ has at least as many i's as it does $i+1$'s.

Note:
- since weight counts number of appearance of $i \in [n]$. it's clear that there can only be a skew tab. T of shape λ/μ with weight ν only if $|\lambda| = |\mu| + |\nu|$. so otherwise $c_{\mu\nu}^{\lambda} = 0$.
- $[n]$ can be chosen for any sufficiently large n. since if the word(T) is Yamanouchi. it can only involve entries from $[k]$. k = # of rows in λ/μ.

Prop 8.10 For n sufficiently large, we have $B_{\lambda/\mu} \cong \bigoplus_{\nu} B_\nu^{\oplus c^{\lambda}_{\mu\nu}}$

Proof # of connected components of $B_{\lambda/\mu}$ iso. to B_ν = # of h.w. elts of this weight ν.

By prop 8.2 (h.w. $\Leftrightarrow$ word(T) is Yamanouchi) $\overset{\nu}{=}$ # of tableaux with word(T) Yamanouchi □

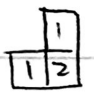
Recall: character of B_λ is a polynomial s_λ, Schur poly. $s_\lambda(t) = \sum_T t^{wt(T)}$ → Laurent polys

we can generalize this. to get character $s_{\lambda/\mu}$ of $B_{\lambda/\mu}$ as a symmetric polynomial

skew Schur functions $s_{\lambda/\mu} = \sum_\nu c^{\lambda}_{\mu\nu} s_\nu$.

Examples

① $\lambda/\mu = (2,2)/(1)$. unique (1 copy) Yamanouchi word $(1,2,1)$. realized as tableau

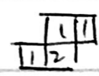


Thus $B_{(2,2)/(1)} \cong B_{(2,1)}$.

$s_{(2,2)/(1)} = s_{(2,1)}$

Embedded via RR into $B^{\otimes 3}$. gives a copy of subcrystal in $B^{\otimes 3}$ iso. $B_{(2,1)}$.

and is the second one in figure 3.3. in textbook.

② $\lambda/\mu = (3,2)/(1)$. Two tableaux  and  with Yamanouchi words.

so $s_{(3,2)/(1)} = s_{(3,1)} + s_{(2,2)}$.