Lecture 4 Wednesday Sept 11

Given root system Φ in Eucl space V

Assume Φ is semi simple, so Φ spans V

Recall

$$\Lambda_{root} = \mathbb{Z}\Phi$$

is a weight lattice for Φ.

Define

$$\Lambda_{sc} = \sum_{i \in I} \mathbb{Z}\varpi_i$$

This a weight lattice of Φ, said to have

<u>simply connected type</u>.

Any weight lattice Λ for Φ satisfies

$$\Lambda_{sc} \supseteq \Lambda \supseteq \Lambda_{root}$$

<u>Problem</u> Assume $\dim V = 2$. For each case of Φ find Λ_{root} and Λ_{sc}

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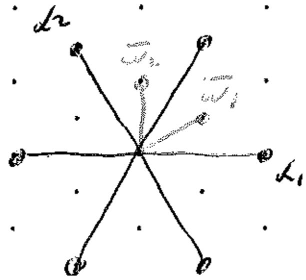
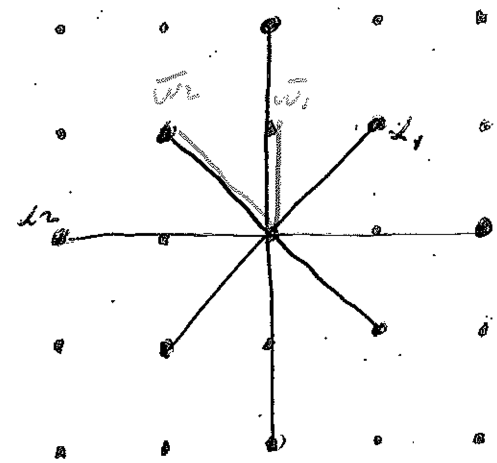
Fundamental weights.



Fundament weights



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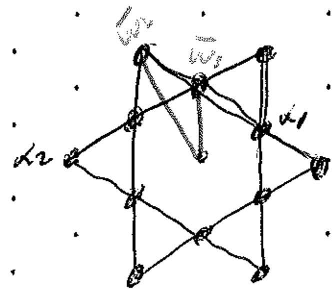
Fundamental weights



No longer assume Φ is semi simple.

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$ "Weyl vector"

LEM 11 We have

(i) $s_i(\rho) = \rho - \alpha_i$ $\quad i \in I$

(ii) $\langle \rho, \alpha_i^\vee \rangle = 1$ $\quad i \in I$

(iii) $\rho - \sum_{j \in I} \varpi_j$ orthog Φ

(iv) Assume Φ is semi simple. Then

$$\rho = \sum_{j \in I} \varpi_j$$

pf (i) Write

$$\rho - \frac{\alpha_i}{2} = \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \alpha$$

s_i permutes summands in RHS

$$s_i\left(\rho - \frac{\alpha_i}{2}\right) = \rho - \frac{\alpha_i}{2}$$

$$\shortparallel$$

$$s_i(\rho) + \frac{\alpha_i}{2}$$

(ii) By (i) and

$$s_i(\rho) = \rho - \langle \rho, \alpha_i^\vee \rangle \alpha_i$$

(iii) By (ii) and

$$\left\langle \sum_{j \in I} \varpi_j , \alpha_i^\vee \right\rangle = \sum_{j \in I} \underbrace{\langle \varpi_j , \alpha_i^\vee \rangle}_{\shortparallel \atop \delta_{ij}}$$

$$= 1$$

(iv) By (iii) and since Φ spans V. $\square$

The simple root systems are classified, and called

$A_r \ (r \geq 1)$, $B_r \ (r \geq 2)$, $C_r \ (r \geq 2)$, $D_r \ (r \geq 4)$,

E_6, E_7, E_8, F_4, G_2.

We now describe these along with some weight Lattices.

In what follows $\{e_i\}_{i=1}^n$ is standard basis for Euclidean space $\mathbb{R}^n$.

A_r type $GL(r+1)$

$V = \mathbb{R}^{r+1}$

$\Phi = \{ e_i - e_j \mid 1 \le i, j \le r+1 \quad i \neq j \}$

$\langle \alpha, \alpha \rangle = 2 \quad \forall \alpha \in \Phi$ $\qquad \Phi$ simply laced

$\alpha^\vee = \alpha \quad \forall \alpha \in \Phi$

$\alpha_i = e_i - e_{i+1} \qquad (1 \le i \le r)$

A root $e_i - e_j$ is positive iff $i < j$

$V = \mathbb{R}\Phi + \mathbb{R}\delta$ $\qquad$ (orthog dir sum)

$\delta = e_1 + e_2 + \cdots + e_{r+1}$

Φ not semisimple

$\mathbb{Z}\Phi = \left\{ \sum_{i=1}^{r+1} a_i e_i \mid a_i \in \mathbb{Z}, \quad \sum_{i=1}^{r+1} a_i = 0 \right\}$

Take

$$\Lambda = \sum_{i=1}^{r+1} \mathbb{Z} e_i = \mathbb{Z}\Phi + \mathbb{Z} e_1$$

For $\alpha = e_i - e_j \in \Phi$

r_α sends
$e_i \leftrightarrow e_j$ $\quad e_k \to e_k \quad k \neq i, j$

For $1 \le i \le r$ $\quad s_i$ sends
$e_i \leftrightarrow e_{i+1}$ $\quad e_k \to e_k \quad k \neq i, i+1$

$W = S_{r+1}$ permutes $\{e_1, e_2, \dots, e_{r+1}\}$

Take $\bar{\omega}_i = e_1 + e_2 + \dots + e_i$ $\quad (1 \le i \le r)$

Weyl vector
$$\rho = \sum_{i=1}^{r+1} \left(\frac{r}{2} - i + 1 \right) e_i$$

For $\lambda = \sum_{i=1}^{r+1} \lambda_i e_i \in \Lambda$

$\lambda \succeq 0$ iff

$\lambda_1 \ge 0$
$\lambda_1 + \lambda_2 \ge 0$
$\vdots$
$\lambda_1 + \lambda_2 + \dots + \lambda_r \ge 0$
$\lambda_1 + \lambda_2 + \dots + \lambda_{r+1} = 0$

$\lambda \in \Lambda^+$ iff

$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{r+1}$

A_r type $SL(r+1)$

$\delta = e_1 + \cdots + e_{r+1}$

Define $V =$ quotient v.s. $\mathbb{R}^{r+1} / \mathbb{R}\delta$

V is Eucl space with

$\langle u + \alpha\delta, v + \beta\delta \rangle = \langle u, v \rangle$

Apply quotient map $\mathbb{R}^{r+1} \to V$ to the root system and wt lattice of type $GL(r+1)$, to get the root system and wt lattice of type $SL(r+1)$.

To be concrete, we identify V with $\delta^\perp$ in $\mathbb{R}^{r+1}$

So

$$V = \left\{ \sum_{i=1}^{r+1} \lambda_i e_i \;\middle|\; \lambda_i \in \mathbb{R}, \quad \lambda_1 + \cdots + \lambda_{r+1} = 0 \right\}$$

$V = \mathbb{R}\Phi$

Φ is semisimple

Define

$$\hat{e}_i = e_i - \frac{\delta}{r+1} \qquad 1 \le i \le r+1$$

Obs $\hat{e}_1 + \cdots + \hat{e}_{r+1} = 0$

$\{\hat{e}_i\}_{i=1}^{r}$ is basis for V

For $\lambda \in V$

write
$$\lambda = \sum_{i=1}^{r} a_i \hat{e}_i \qquad a_i \in \mathbb{R}$$

Write
$$\lambda = \sum_{i=1}^{r+1} \lambda_i e_i \qquad \lambda_i \in \mathbb{R} \qquad \lambda_1 + \cdots + \lambda_{r+1} = 0$$

The $\{a_i\}_{i=1}^{r}$, $\{\lambda_i\}_{i=1}^{r+1}$ are related by

$$\lambda_i = a_i - \frac{a_1 + a_2 + \cdots + a_r}{r+1} \qquad 1 \le i \le r$$

$$\lambda_{r+1} = - \frac{a_1 + a_2 + \cdots + a_r}{r+1}$$

and
$$a_i = \lambda_i - \lambda_{r+1} \qquad 1 \le i \le r$$

We have
$$\mathbb{Z}\Phi = \left\{ \sum_{i=1}^{r+1} \lambda_i e_i \;/\; \lambda_i \in \mathbb{Z}, \quad \lambda_1 + \cdots + \lambda_{r+1} = 0 \right\}$$
$$= \left\{ \sum_{i=1}^{r} a_i \hat{e}_i \;/\; a_i \in \mathbb{Z}, \quad a_1 + \cdots + a_r \in (r+1)\mathbb{Z} \right\}$$

We have
$$\Lambda = \sum_{i=1}^{r+1} \mathbb{Z}\hat{e}_i = \sum_{i=1}^{r} \mathbb{Z}\hat{e}_i$$
$$= \left\{ \sum_{i=1}^{r+1} \lambda_i e_i \;/\; \lambda_k - \lambda_j \in \mathbb{Z}, \quad \lambda_1 + \cdots + \lambda_{r+1} = 0 \right\}$$

Group hom

$$\Lambda \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/(r+1)\mathbb{Z}$$

$$\sum_{i=1}^{r} a_i \hat{e}_i \longrightarrow \sum_{i=1}^{r} a_i \quad \text{quot}$$

is surj with kernel $\mathbb{Z}\Phi$

so $\mathbb{Z}\Phi$ has index $r+1$ in Λ

For $\alpha = e_i - e_j \in \Phi$

r_α sends

$\hat{e}_i \leftrightarrow \hat{e}_j \qquad \hat{e}_k \to \hat{e}_k \quad k \neq i, j$

$W = S_{r+1}$ permutes $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{r+1}\}$

For $1 \leq i \leq r$ the ith fundamental wt is

$$\varpi_i = \hat{e}_1 + \hat{e}_2 + \cdots + \hat{e}_i$$
$$= e_1 + e_2 + \cdots + e_i - \frac{i\,\delta}{r+1}$$

We have

$$\Lambda_{sc} = \sum_{i=1}^{r} \mathbb{Z}\varpi_i = \sum_{i=1}^{r} \mathbb{Z}\hat{e}_i = \Lambda$$

Fn $\lambda \in \Lambda$ write

$$\lambda = \sum_{i=1}^{r} a_i \hat{e}_i = \sum_{i=1}^{r+1} \lambda_i e_i \qquad (\lambda_1 + \dots + \lambda_{r+1} = 0)$$

$\lambda \geq 0$

iff

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \geq 0 \qquad 1 \leq i \leq r$$

iff

$$(r-i+1)(a_1 + \dots + a_i) \geq i(a_{i+1} + \dots + a_r) \qquad 1 \leq i \leq r$$

Also $\lambda \in \Lambda^+$

iff

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r+1}$$

iff $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$

B_r type spin $(2r+1)$

$V = \mathbb{R}^r$

$$\Phi = \{ \pm e_i \pm e_j \mid 1 \le i < j \le r \} \cup \{ \pm e_i \mid 1 \le i \le r \}$$

Φ semi simple

For $\alpha = \pm e_i \pm e_j$ $\quad \alpha^\vee = \alpha \quad \langle \alpha, \alpha \rangle = 2$

$\alpha = \mp e_i$ $\quad \alpha^\vee = 2\alpha \quad \langle \alpha, \alpha \rangle = 1$

Φ not simply laced

Take

$\alpha_i = e_i - e_{i+1} \quad 1 \le i \le r-1$

$\alpha_r = e_r$

$$\Phi^+ = \{ e_i \pm e_j \mid 1 \le i < j \le r \} \cup \{ e_i \mid 1 \le i \le r \}$$

$$\mathbb{Z}\Phi = \sum_{i=1}^{r} \mathbb{Z} e_i$$

"orthogonal weights" "spin weights"

Take

$$\Lambda = \mathbb{Z}\Phi \cup \left\{ \tfrac{1}{2}\sum_{i=1}^{r} a_i e_i \;\middle|\; a_i \in \mathbb{Z}_{odd} \quad 1 \le i \le r \right\}$$

$$= \left\{ \tfrac{1}{2}\sum_{i=1}^{r} a_i e_i \;\middle|\; a_1, a_2, \dots, a_r \text{ integers with same parity} \right\}$$

$\mathbb{Z}\Phi$ index 2 in Λ

α	r_α sends		
$e_i - e_j$	$e_i \leftrightarrow e_j$	$e_k \to e_k$	$k \neq i, j$
$\pm(e_i + e_j)$	$e_i \leftrightarrow -e_j$	$e_k \to e_k$	$k \neq i, j$
$\pm e_i$	$e_i \to -e_i$	$e_k \to e_k$	$k \neq i$

For $1 \le i \le r-1$ $\quad s_i$ sends

$e_i \leftrightarrow e_{i+1} \qquad e_k \to e_k \quad k \neq i, i+1$

s_r sends

$e_r \to -e_r \qquad e_k \to e_k \quad 1 \le k \le r-1$

$W = (\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r$ acts on the set

$\{ \pm e_1, \pm e_2, \dots, \pm e_r \}$

by perms and sign changes

For $1 \le i \le r$

$$\overline{\omega}_i = \begin{cases} e_1 + \cdots + e_i & 1 \le i \le r-1 \\ \dfrac{e_1 + \cdots + e_r}{2} & i = r \end{cases}$$

$$\Lambda_{sc} = \sum_{i=1}^{r} \mathbb{Z}\,\overline{\omega}_i = \Lambda$$

Weyl vector

$$\rho = \sum_{i=1}^{r} (r - i + \tfrac{1}{2}) e_i$$

For $\lambda = \sum_{i=1}^{r} \lambda_i e_i \in \Lambda$,

$\lambda \geq 0$ iff

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \ge 0 \qquad 1 \le i \le r$$

$\lambda \in \Lambda^+$ iff

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0.$$