Math 846

Lecture notes for presentation.

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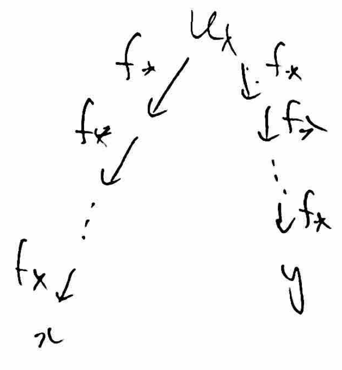
*. I presented Thm 8.7 on Dec 4th.

Thm 8.7 Suppose $x, y \in B^{\otimes k}$. Then $Q(x) = Q(y)$ if and only if x and y lie in the same connected subcrystal.

Pf) (i) "x and y lie in the same connected subcrystal" $\Rightarrow$ "$Q(x) = Q(y)$"

Suppose x and y are in the same connected subcrystal C.

Let u_λ be the highest weight elt of C.



- Every elt of C is obtained from u_λ by applying f_j in some sequence.

$\Rightarrow$ So it suffices to show $Q(x) = Q(y)$ if $y = f_j x$ for some $j \in I$.

Let $x = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k}$

(Then $P(x) = (\phi \leftarrow i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_k)$: insertion tableau.

$Q(x)$: recording tableau of $P(x)$)

recall the signature rule §2.4, pg 22.
(to compute $f_j x$)

Case ⓐ $i_t = j$ or $j+1$ for all t.

Let's say f_j acts on $\boxed{i_m}$
(which means i_m is the rightmost unbracketed j.)

$$\underbrace{i_1 \cdots}_{\text{no unpaired } j+1} \quad \underset{j}{\overset{i_m}{\shortparallel}} \quad \underbrace{\cdots\quad i_t}_{\text{no unpaired } j}$$

(Ex: $x = [j+1, j], [j+1, j], \underset{i_m}{j}, [j+1, j], j+1,$

Then $y = f_j x = [j+1, j], [j+1, j], \underset{m\text{-th}}{j+1}, [j+1, j], j+1$.)

Consider initial segment $\{i_1, i_2, \cdots, i_{m-1}, i_m\}$.
Every $j+1$ in this is paired with a later j.
All such $j+1$ will be bumped by insertion of later j.
Replacing $i_m = j$ by $j+1$ will not change this.
$\Rightarrow$ Up to m, $Q(y)$ equals to $Q(x)$

* what happens after i_m?

Note that i_m is the rightmost unpaired j..

When i_m is j, i_m is not bumped by $i_{m+1}, --, i_k$.

since $i_m \leq i_t$ for $t = m+1, --, k$.

After replacing $i_m = j$ by $j+1$,

$i_m = j+1$ is not bumped by $i_{m+1}, --, i_k$.

since every j right to i_m is already paired with $j+1$ which lies to the left of the j.

Therefore, $Q(x) = Q(y)$.

Case (b) general case. (i_t does not have to be j or $j+1$.)

$x = \boxed{i_1} \otimes \cdots \otimes \boxed{i_k}$

let $I_{j,j+1}$ be the subsequence of $I := (i_1, --, i_k)$ in which the entries are equal to j or $j+1$.

Write $I_{j,j+1} := (i_{s_1}, i_{s_2}, --, i_{s_\ell})$

let's say f_j acts on i_{s_m} $(= j)$

subsequence

$I_{\bar{j},j+1} = \quad \bar{\lambda}_{s_1} \quad \bar{\lambda}_{s_2} \; \cdots \quad \bar{\lambda}_{s_m} \; \cdots \quad \bar{\lambda}_{s_l}.$

$\bar{\lambda}_{s_m} = \bar{j}$

no unpaired $j+1$ (before $\bar{\lambda}_{s_m}$)

the rightmost unpaired $\bar{j}$.

no unpaired $\bar{j}$ (after $\bar{\lambda}_{s_m}$)

$I = \quad \bar{\lambda}_1, \bar{\lambda}_2 \; \cdots \quad \bar{\lambda}_{s_m} \; \cdots \quad \bar{\lambda}_k.$

$\bar{\lambda}_{s_m} = \bar{j}$

no unpaired $j+1$.

the rightmost unpaired $\bar{j}$.

no unpaired $\bar{j}$

Consider $\{\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_{s_m}\}$

Every $j+1$ in this is paired with a later $\bar{j}$.

$\Rightarrow$ All such $j+1$ will be bumped by insertion of later p. where $p \leq j$.

(this is because there could be some number less than $\bar{j}$ between $j+1$ and $\bar{j}$.)

The point is these $j+1$ are bumped by $\bar{\lambda}_t$ where $t < s_m$.

Hence, replacing $\bar{\lambda}_{s_m} = \bar{j}$ by $j+1$ will not change this

$\Rightarrow$ up to s_m, $Q(y)$ equals to $Q(x)$

* What happens after i_{s_m}?

Note that i_{s_m} is the right most unpaired $\bar{j}$.

When i_{s_m} is $\bar{j}$, i_{s_m} is not bumped by $i_{s_m+1}, \cdots, i_{s_\ell}$.

After replacing $\bar{i}_{s_m} = \bar{j}$ by $j+1$,

$i_{s_m} = j+1$ is not bumped by $i_{s_m+1}, \cdots, i_{s_\ell}$.

Since every $\bar{j}$ right to i_{s_m} is already paired with $j+1$ which lies to the left of the $\bar{j}$,

which implies every $j+1$ after $\bar{i}_{s_m}$ is bumped by some $\bar{i}_t$ where $\bar{i}_t \le \bar{j}$.

Therefore, replacing $i_{s_m} = \bar{j}$ by $j+1$ will not change insertions and bumpings after s_m.

$\therefore \quad Q(x) = Q(y)$

(ii) "$Q(x) = Q(y)$" $\Rightarrow$ "x and y lie in the same connected subcrystal"

Suppose $Q(x) = Q(y)$

let $x \in C_1$, $y \in C_2$.

let $\begin{cases} u_{\lambda_1} \text{ be the highest weight vector of } C_1 \\ u_{\lambda_2} \quad \text{"} \quad \text{"} \quad C_2 \end{cases}$

Then $\begin{cases} Q(u_{\lambda_1}) = Q(x) \\ Q(u_{\lambda_2}) = Q(y) \end{cases}$, since Q is constant on connected subcrystal.

since $Q(x) = Q(y)$, $\Rightarrow Q(u_{\lambda_1}) = Q(u_{\lambda_2})$

Note that if λ is the shape of Q, then P must be the highest weight vector in B_λ.

Hence, $Q(u_{\lambda_1}) = Q(u_{\lambda_2})$ implies $P(u_{\lambda_1}) = P(u_{\lambda_2})$.

since x is completely determined by $P(x)$ and $P(y)$, we get $u_{\lambda_1} = u_{\lambda_2}$, which implies $C_1 = C_2$ because of the uniqueness of the highest weight vector.

$\Rightarrow$ x and y lie in the same connected subcrystal