Crystals and Schensted Insertion

We'll explain a relationship between Schensted insertion and crystals.

Def $\mathcal{C}, \mathcal{D}$ two crystals. their direct sum $\mathcal{C} \oplus \mathcal{D}$ is the disjoint union of $\mathcal{C}$ and $\mathcal{D}$.

In the following, B_λ is the $GL(n)$ crystals of highest weight λ. $\mathbb{B}$ is the standard $GL(n)$ crystal.

Prop We have an isomorphism $\mathcal{B}_{(k)} \otimes \mathbb{B} \simeq B_{(k+1)} \oplus B_{(k,1)}$
The correspondence is give by $T \otimes j \longmapsto T \leftarrow j$
Thus, $T \otimes j \equiv T \leftarrow j$ (plactically equivalent).

Proof First, we show that $\mathcal{B}_{(k)} \otimes \mathbb{B} \simeq B_{(k+1)} \oplus \mathcal{B}_{(k,1)}$.
Since $B_{(k)} \otimes \mathbb{B}$ is Stembridge, $B_{(k)} \otimes \mathbb{B}$ is a disjoint union of crystals, isomorphic to crystals of tableaux.
Each crystal of tableaux has a unique highest weight, so we can determine $\mathcal{B}_{(k)} \otimes \mathbb{B}$ by finding all highest weight vectors :

$$T \otimes j \quad \text{s.t.} \quad \varepsilon_i(T \otimes j) = 0 \quad \forall i = 1, 2, \cdots, n.$$

Since

$$\varepsilon_i(T \otimes j) = \max\left(\varepsilon_i(j),\ \varepsilon_i(T) + \varepsilon_i(j) - \varphi_i(j) \right) = 0.$$

$$\varepsilon_i(j) = 0 \quad \forall i \quad \Rightarrow \quad j = 1.$$

We have
$$\varphi_i(1) = \begin{cases} 1, & i = 1 \\ 0, & i = 2, 3, \cdots, n. \end{cases}$$

So
$$\varepsilon_i(T) = \begin{cases} 0 \text{ or } 1, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

$\Rightarrow \quad T = 1 \cdots 1 \quad \text{or} \quad 1 \cdots 12$

So $B(k) \otimes \mathbb{B}$ contains 2 highest weight vectors:

$$1 \cdots 1 \otimes 1 \quad \text{and} \quad 1 \cdots 12 \otimes 1 .$$

Their weights are $(k+1, 0, \cdots, 0)$ and $(k, 1, 0, \cdots, 0)$

Hence $B(k) \otimes \mathbb{B} \simeq \mathcal{B}(k+1) \oplus \mathcal{B}(k,1)$.

Next, we show in this isomorphism. $T \otimes j$ corresponds to $T \leftarrow j$.

Let $T = i_1 \cdots i_k$.

If $j \geqslant i_k$, then

$$i_1 \cdots i_k \otimes j = i_1 \otimes \cdots \otimes i_k \otimes j = i_1 \cdots i_k j = T \leftarrow j.$$

So the elements $T \otimes j$ with $j \geqslant i_k$ correspond to eles of $\mathcal{B}(k+1)$.

If $j < i_k$, already know this corresponds to an ele of $B(k,1)$.

"Knuth equiv $\Rightarrow$ plactically equiv", so enough to show their reading words are Knuth equivalent:

$$W(T \otimes j) \equiv_K W(T \leftarrow j).$$

Let $s \leqslant k$ the smallest integer s.t. $j < i_s$ then i_s is bumped and

$$T \leftarrow j = \begin{array}{l} i_1 \cdots i_{s-1}\, j\, i_{s+1} \cdots i_k \\ i_s \end{array}$$

Want $i_1 i_2 \cdots i_k j \equiv_K i_s i_1 \cdots i_{s-1} j\, i_{s+1} \cdots i_k$.

We just do a typical example. Suppose

$$T = 1123334 \quad j = 2.$$

$$T \leftarrow j = \begin{array}{l} 1122334 \\ 3 \end{array}$$

Want $11233342 \equiv_K 31122334$.

$a < b \leqslant c$, $bca \equiv_K bac$: $\quad 11233342 \equiv_K 11233324 \equiv_K 11233234$

$\equiv_K 11232334 \equiv_K 11322334 \equiv_K 13122334$

$\equiv_K 31122334$. $\square$

$a \leqslant b < c$
$cab \equiv_K acb$

Recall P. Q in the RSK algorithm.

If $x = i_1 \otimes \cdots \otimes i_k$,

$P(x) = \emptyset \leftarrow i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_k$.

$Q(x)$ is the recording tableau, i.e. an entry r is in the location of the new box of $\emptyset \leftarrow i_1 \leftarrow \cdots \leftarrow i_r$.

Example $x = 3 \otimes 1 \otimes 2 \otimes 1 \otimes 4$

$$P(x) = \begin{matrix} 1 & 1 & 4 \\ 2 & & \\ 3 & & \end{matrix} \qquad 3,\ \begin{matrix} 1 \\ 3 \end{matrix},\ \begin{matrix} 1 & 2 \\ 3 & \end{matrix},\ \begin{matrix} 1 & 1 \\ 2 & \\ 3 & \end{matrix},\ \begin{matrix} 1 & 1 & 4 \\ 2 & & \\ 3 & & \end{matrix}$$

$$Q(x) = \begin{matrix} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{matrix}$$

Next thm will show P determines x up to plactical equivalence. (Q determines the connected component).

Thm $\mathbb{B}^{\otimes k}$ decomposes into a disjoint union of crystals, each isomorphic to B_λ. λ is a partition of k of length $\leq n$.

Let $x \in \mathbb{B}^{\otimes k}$.

(i) $x \equiv P(x)$

(ii) If λ is the shape of $P(x)$ and $Q(x)$, then x lies in a subcrystal isomorphic to B_λ.

(iii) If $P(x) = P(y)$, then $x \equiv y$.

Proof (ii) and (iii) follows from (i). Enough to show Part (i).

First we show

$$T \otimes j \equiv T \leftarrow j.$$

If T is a single row, this is just the previous proposition.

If general, say T has r rows $T_1, \cdots, T_r$.
By the prop. $T_1 \otimes j \equiv T_1 \leftarrow j$.
If $T_1 \leftarrow j$ is a single row, then $T \leftarrow j \equiv T_r \otimes \cdots \otimes T_1 \otimes j \equiv T \otimes j$.
If j bumps some j', let k_1 be the length of T_1 then
$T_1 \leftarrow j$ has shape $(k_1, 1)$ and $T_1 \leftarrow j \equiv j' \otimes T_1' \in B(k_1, 1)$
T_1' is T_1 with j' replaced by j. Now

$$T \otimes j \equiv T_r \otimes \cdots \otimes T_1 \otimes j \equiv T_r \otimes \cdots \otimes T_2 \otimes j' \otimes T_1$$

We repeat this process until we are done.
Now show $x \equiv P(x)$ by induction. Say $x = i_1 \otimes \cdots \otimes i_k$
When x is in $\mathbb{B}^{\otimes 1}$, trivial. $(k=1)$
By inductive hypothesis,

$$i_1 \otimes \cdots \otimes i_{k-1} \equiv T$$

where $T = \emptyset \leftarrow i_1 \leftarrow \cdots \leftarrow i_{k-1}$
So $x \equiv T \otimes j \equiv T \leftarrow j = P(x)$. $\square$