MATH 846 PRESENTATION

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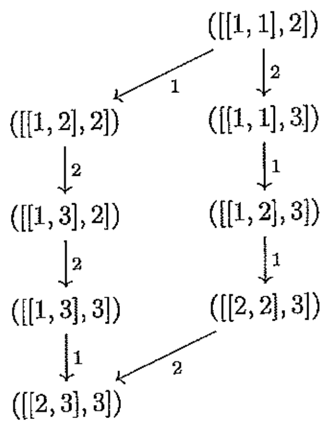
1 Introduction

So for this talk we will begin to consider something called the *plactic monoid.* Recall that a 'monoid' is merely a group without inverses – a set with an identity and an associative binary operation.

If B is the standard, A_r, $GL(n)$ crystal, let's recall what this looks like. It is a simple chain of numbers $1 \xrightarrow{1} 2 \xrightarrow{2} ...(n-1) \xrightarrow{n-1} n$.

Let λ be a partition of k with at most n elements; recall that we can form a crystal out of partitions.

For instance, this is what the $B_{(2,1)}$ crystal looks like:



The 1 arrow (denoting the f_1 operation) finds a 1 inside the crystal that we can change to a 2 while keeping it semistandard and does so. Likewise for the two.

This can be seen by use of the signature rule: for instance, recall from row read that

$$([[1,2],3]) = \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

By using the signature, rule, look at subscript 1. The last unpaired closing parenthesis is at 1; this represents the 1 we move (and it gets us to $([[2,2],3])$). The first unpaired opening parenthesis is at 2 (and it gets us to $([[1,1],3])$) representing e_1 (which is the reverse move). Also, if we look at f_2 and e_2 all parentheses (opening at 3 and closing at 2) are paired so they both go to the special symbol $\varnothing$.

2 What is the plactic monoid?

If C_1 and C_2 are crystals of the same type, then if x_1 and x_2 are elements of these crystals, then if the connected components containing them are isomorphic we shall consider x_1 and x_2 to be plactically equivalent. Denote this by $x_1 =_P x_2$. Since this connects through tensor products, the *plactic monoid* is the monoid of equivalence classes, where the identity is the empty crystal and the operation is tensoring (which is associative).

This allows us to rephrase one of our best known results, and that is that each CC (connected component) of the k-th power tensor product of our crystal B is isomorphic to some partition (and there is at least one isomorphic to every partition). If we go back over the results from chapter 3, we can now define a monoid on B^k (the k-dimensional tensor product) by using the equivalencies as needed.

3 Knuth Equivalences

If we have an element of the tensor product as a product of K squares of the form $u_1, u_2, ..., u_k$, consider Knuth equivalences, essentially saying that $b \times a \times c$ can be switched if c is larger than b and a is not between them (the last two can be swapped if $c \geq b > a$ and the first two can be swapped if $a > c \geq b$.

We can do this at any point in our sequence, and in either direction. For example if we have

$$\boxed{4} \otimes \boxed{8} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{5}$$

is equivalent, by the above,

$$\boxed{4} \otimes \boxed{8} \otimes \boxed{2} \otimes \boxed{5} \otimes \boxed{1}$$

, and then, since 5 is (strictly) between 2 and 8, it is equivalent to

$$\boxed{4} \otimes \boxed{2} \otimes \boxed{8} \otimes \boxed{5} \otimes \boxed{1}$$

.

FYI: It turns out that standard tableaux of shape λ are in 1-1 correspondence with the associatively shaped subtableaus of B^k. For instance, with 3 we got the scenario that there was one of type $(1,1,1)$, one of type (3), and two of type $(2,1)$ (as there are 2 standard "L-shaped" tableaux). If we continually insert numbers we can generate a "canonical" tableau. It so happens that x is represented by that specific tableau, and the whole CC by Q (which is the "recording tableau", and is standard)

How many such connected components are there? Well, we'll get to that later.

Can we reverse this process and turn tensor product into tableau? We can with Schensted insertion. let's go back to our old tactic and consider the reading word $1 \times 2 \times 2 \times 1 \times 3 \times 2$.

We create the following sequence of additions. Start with the empty tableau (shape $\emptyset$). Add the 1 (shape (1)). Add the 2, which we can do without much issue (shape (2)). Add the other 2, doing this without any difficulty (shape (3)). Add the 1, which bumps our first 2 into the second row (shape (3,1)). Add the 3 to the first row (4,1) and then add the 2 to the first row once again, and the 3 falls into the second row, leading to $P = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ and a Q of $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2}\end{array}$. This allows us to "multiply" two partitions of n into a partition of size $2n$.

So let's denote a "word" as a sequence of numbers. For instance, 2514 represents $\boxed{2} \otimes \boxed{5} \otimes \boxed{1} \otimes \boxed{4}$.

4 Yamanouchi words

A Yamanouchi word is one whose final segments all contain weakly more is than $i+1$s for all i. For example,

$$77361545526544444444432323232111111112323232311111111121$$

is a Yamanouchi word.

PROPOSITION:

Yamanouchi words correspond to highest weights elements.

Proof. Recall that an element is highest-weight if $e_i(x) = \varnothing$ for all i, or alternatively that $\epsilon_i(x) = 0$ for all i.

We will use the signature rule. This, converted) into actual math, gives us the following equation:

$$\epsilon_i(x) = \max_{j=1}^{k}(\sum_{h=j}^{k} \epsilon_i(u_h) - \sum_{h=j+1}^{k} \phi_i(u_h))$$

.

This is a consequence of Lemma 2.33 from the textbook.

Should the maximum not be zero it is obtained when $u_j = i+1$ (as this is 1 less than when $u_j = i$; this can be expanded; in this case the maximum is 1 as the ϵ_is sum to 1 but the ϕ_is sum to 0). Then we can add $\phi_i(u_{i+1})$ (which is 0), or rather $\phi_i(u_j)$, to the second sum. This allows us to define the equation as the number of is in final segment minus the number of $i+1$s, so if this max is indeed zero (i.e. if $\epsilon_i = 0$) it requires that we have a Yamanouchi word. □

Not every element is in the Row Read embedding of B_λ, but they're all in a component isomorphic to the lambda crystal.

It so happens that there are 4 Connected Components of B^3, as above. This is, as similarly, there are 2 Yamanouchi words with $(1,1,2)$, namely 121 and 211 (since 112 isn't a Yamanouchi word), corresponding to 2 highest weight elements and thus 2 connected components for the $(2,1)$ tableau shape. There are also individual Yamanouchi words (111 and 321) corresponding to the other two tableau shapes (The first is (3) and the second is $(1,1,1)$; recall that columns are strictly increasing).) Consider $B_{111}, B_{21}, B'_{21}, B_3$ as our four components.

Now we shall go over the proof that if $x, y \in \mathbb{B}^{\otimes k}$, then if their words are Knuth-equivalent then $x =_P y$. Note that this is a one-sided proof.

5 The Big Proof

This can be restricted to the case but $k = 3$ but can be generalized for larger powers. Here's how we do it:

Let's consider when $k = 3$. Assume the premise.

By considering tableaux, $a \times b \times c$ is in $B_{(3)}$ if $a \le b \le c$, $B_{1,1,1}$ if $a > b > c$ (recall that columns are strictly increasing), B_{21} if b is the smallest (perhaps tied with c), B'_{21} otherwise. Consider the isomorphism between the two equishapic crystals. I acknowledge that that isn't a word. Suppose $a \leq b < c$.

It so happens that $x = (c, a, b)$ and $y = (a, c, b)$ are Knuth-Equivalent. But are they isomorphic? Induct on b. Let $b = a$, so x is the unique element with its corresponding weight. We're checking isomorphism of (c, a, a) and (a, c, a); they have the same weight $(2wt(a)+wt(c))$ and they appear in connected components of the same type so such an isomorphism exists.

Status: $a \leq b < c$.

Assumption: $\theta(x_1) = y_1$ where $x_1 = \boxed{c}\otimes\boxed{a}\otimes\boxed{b-1}$ and $y_1 = a \times c \times b-1$. So assume it works for case minus one. Next, we will apply f_{b-1} to this.

What happens? It can be checked with the signature rule that f_{b-1} applied to x_1 is indeed x.

To be fair, it makes a fair amount of sense. (Recall that ϕ_{b-1} is, of course, since our crystal is seminormal, the number of f_{b-1}s required to reach the special symbol, and so, since $e_{b-1}(c)$ is the special symbol we can carry through $f_{b-1}(x_1)$ as $c \times f_{b-1}(a, b-1) = x$. Rigorously it is because $\phi_{b-1}(a \otimes (b-1)) > \epsilon_{b-1}(c)$ as the latter is zero.

A similar argument shows that $f_{b-1}(y_1) = a \times c \times f_{b-1}(b-1) = y$ (noting that $\epsilon_{b-1}(a \vee c)$ is zero, and $\phi_{b-1}(b-1) > 0$)

So recalling that θ is the isomorphism map between the two connected components,

$$\theta(x) = \theta(f_{b-1}(x_1)) = f_{b-1}\theta(x_1) = f_{b-1}y_1 = y$$

For the other half of the proof (proving if $a < b \leq c \rightarrow b \times a \times c = b \times c \times a$) the proof is quite similar and will be omitted.

6 Extension

To extend to words of length > 3, we can write $x = u \otimes x_1 \otimes v$ and $y = u \otimes y_1 \otimes v$. x_1 and y_1 are length 3 and Knuth equivalent and thus plactically equivalent – isomorphism of subcrystals C and D of B^3 contianing them, leads to isomorphism of $B^l \times C \times B^m$ to $B^l \times D \times B^m$. It must take x to y for obvious reasons (as they are by construction identical except for these three numbers), so the theorem has been proven

7 A few further notes

Here are few further notes.

Here's a question I came to wonder: the number of strings of length n, in the alphabet from $1 \rightarrow k$, up to Knuth equivalency, is what?

Let's find a pattern.

$sk(1, k) = k$. This should be obvious.

$sk(n, 1) = 1$. This should also be obvious, as it has to be $1, 1, 1, 1, \ldots$.

$sk(2,k) = k^2$. This should still be obvious, as Knuth transformation requires at least 3 elements.

$sk(3,2) = 6$. This can be checked because $121 = 211$ and $212 = 221$.

Claim:

$sk(3,k) = \frac{k(2k^2+1)}{3}$

Proof:

We have three cases.

If we have three distinct numbers, a, b, c, and $a < b < c$, Knuth equivalence tells us that $bac = bca$ and $acb = cab$, so there are 4 equivalence classes for each trio of distinct numbers, leading to $\frac{4k(k-1)(k-2)}{6}$ strings.

If we have two distinct and one other, a, a, b, then if $a > b$, aba and aab are equivalent, while if $b > a$, aba and baa are equivalent. Either way there are 2 equivalence classes for each ordered pairs, for a total of $2k(k-1)$ equivalence classes.

Finally, if all three numbers are the same (a, a, a) there is clearly one string, so there are k equivalence classes, so the total number of equivalence classes is:

$$\frac{2k(k-1)(k-2) + 6k(k-1) + 3k}{3} = \frac{2k^3 - 6k^2 + 4k + 6k^2 - 6k + 3k}{3} = \frac{k(2k^2+1)}{3}$$

Unfortunately, for strings of length 4 or longer it becomes much more complicated as we can often do multiple Knuth-switches. However, we can bound the number of equivalence classes by determining the number of connected components of $\mathbb{B}^{\otimes k}$ corresponding to our tableau shape.

This is directly related to the number of semistandard (minimal) tableaux for a given partition.

8 Hooks

This is given by something called the *hook-length formula.* This formula tells us how many standard tableaux exist of a given shape, and states that the number of such tableau with n squares is $\frac{n}{H}$, where H is the product of the hook-lengths (the hook length of a square is the length of the sequence of squares that goes up from the bottom, turns right at this square, and goes until the right end of the tableau) of all n squares.

For instance, if we look at the tableau of shape $(5, 3, 3, 2)$ then the total hook lengths follow the following pattern:

8	7	5	2	1
5	4	2		
4	3	1		
2	1			

The product of these numbers is 537600, so there are $13!/537600 = \boxed{11,583}$ standard tableaux of this shape

This means that for this partition of the number 13, $\mathbb{B}^{13}$ (assuming B is large enough) will have a staggering 11,583 connected components corresponding to this tableau shape. Perhaps this is why we didn't make an example of thirteenth powers of tableaux.

However, it can be used to generate bounds on things, such as the number of strings that are Knuth equivalent to a given 13-string. If we assume that the above tableau shape has the greatest number of SYTs of any shape of size 13 (I do not actually know if this is true), there are at most 11583 tableaux plactically equivalent / Knuth equivalent to any word, which implies that $sk(13,n) \geq \frac{n^{13}}{11583}$ (for instance, $sk(13,3)$ has to be at least $\frac{1594323}{11583} = 138$). I would like to see if this bound can be improved, but as of now that remains a future investigation.