Lecture 37 Wednesday Nov 27

Recall Dual RSK gives bijection between

(i) set of $k \times n$ matrices with entries $\in \{0,1\}$

(ii) set of ordered pairs (P, Q) such that

- P is s. stand tableau with entries $\in \{1,2,..,n\}$
- Q is s. stand tableau with entries $\in \{1,2,..,k\}$
- P, Q conjugate shape

We described dual RSK via crystals.
Next we describe dual RSK via <u>dual</u> <u>Schensted</u> <u>insertion</u>

Def. Call a tableau T <u>dual</u> <u>semi</u> <u>standard</u>

whenever

- in each row entries strictly inc $\longrightarrow$
- in each col entries weakly inc $\downarrow$

So T is dual semi standard iff T^t is s. standard.

Given dual s. stand tableau T with entries in $\{1, 2, \ldots, n\}$

For $1 \le j \le n$, <u>dual Schensted insertion</u> gives a way to insert an extra box $\boxed{j}$ into T to get another dual s. stand tableau, denoted

$$T \Leftarrow j$$

[compare notations

$T \leftarrow i$ — Schensted insertion

$T \Leftarrow j$ — dual Schensted insertion]

<u>Case</u> T has one row

ex $n=6$

T: 1245

$T \Leftarrow 1$:	$\begin{matrix}1245\\1\end{matrix}$	"1 bumps 1"
$T \Leftarrow 2$:	$\begin{matrix}1245\\2\end{matrix}$	"2 bumps 2"
$T \Leftarrow 3$:	$\begin{matrix}1235\\4\end{matrix}$	"3 bumps 4"
$T \Leftarrow 4$:	$\begin{matrix}1245\\4\end{matrix}$	"4 bumps 4"
$T \Leftarrow 5$:	$\begin{matrix}1245\\5\end{matrix}$	"5 bumps 5"
$T \Leftarrow 6$:	12456	

In summary, for

T : $a_1 a_2 \cdots a_t$ $\qquad a_1 < a_2 < \cdots < a_t$

Case	$T \Leftarrow j$	Check $T \Leftarrow j$ is d. s. stand
$j \le a_1$	$j a_2 \cdots a_t$ a_1	row $j \le a_1 < a_2$ ✓ col $j \le a_1$ ✓
$\exists s$ $(2 \le s \le t)$ st $a_{s-1} < j \le a_s$	$a_1 \cdots a_{s-1}\, j\, a_{s+1} \cdots a_t$ a_s	row $a_{s-1} < j \le a_s < a_{s+1}$ ✓ col $a_1 \le a_{s-1} < a_s$ ✓
$a_t < j$	$a_1 \cdots a_t\, j$	row $a_t < j$ ✓

For multirow T, we proceed as in regular Schensted insertion.

Ex Fa

$$T = \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & \\ 2 & 3 & 5 & \\ 2 & 4 & 5 & \\ 3 & 4 & & \\ 3 & & & \end{array}$$

$$T \Leftarrow 1 = \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & \\ 1 & 3 & 5 & \\ 2 & 4 & 5 & \\ 2 & 4 & & \\ 3 & & & \\ 3 & & & \end{array}$$

<u>Caution</u> In general

$$T \Leftarrow j \neq (T^t \leftarrow j)^t$$

Ex, cont

$$T^t = \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 4 & 4 & \\ 3 & 4 & 5 & 5 & & \\ 5 & & & & & \end{matrix}$$

$$T^t \leftarrow 1 = \begin{matrix} 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 4 & 4 & \\ 3 & 3 & 5 & 5 & & \\ 4 & & & & & \\ 5 & & & & & \end{matrix}$$

$$\neq (T \Leftarrow 1)^t$$

LEM For dual s. stand tableau T and $1 \le j \le n$ Consider $T \Leftarrow j$

Consider any entry in T, say $T(r,k)$

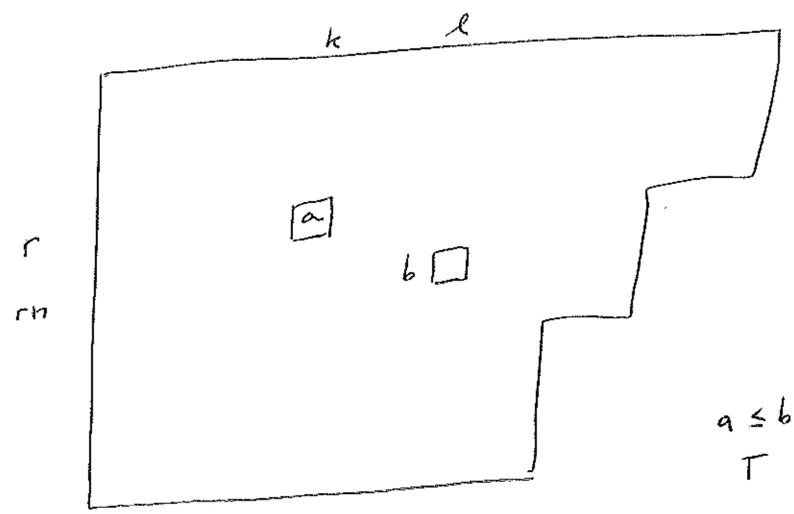
Assume $T(r,k)$ is bumped to row $r+1$ and col l

then $l \le k$

pf Suppose $l > k$ write

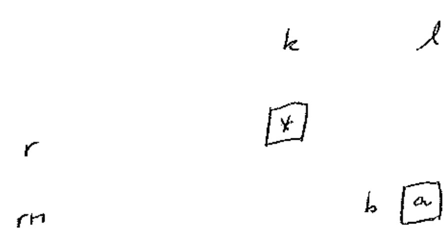
$a = T(r,k)$ $\qquad$ $b = T(r+1, l-1)$

T:



$a \le b$ since T d.s.st

$T \Leftarrow j$:



$b < a$ by bumping constr, cont. $\square$

For dual Schensted insertion

we speak of

bumping path,

new Location,

Landing Location

as before.

Bumping path forms vertical strip by prev lemma.

Entries down bumping path weakly increase

For a dual semi stand tableau T

and $1 \le j \le n$.

Suppose we are given

$T \Leftarrow j$ and new location.

Recover T and j

ex

$$T \Leftarrow j \quad \begin{array}{llll} 1 & 2 & 4 & 6 \\ 1 & 3 & 4 & 6 \\ 2 & 3 & 5 & \\ 2 & 3 & & \\ 3 & \boxed{4} & & \\ 5 & & & \\ 7 & & & \end{array}$$

$\square$ = new loc

4 bumped from row 4 of T

T row 4 is str inc

T row 4: $\quad 2\ 4 \quad \leftarrow 3$

3 bumped from row 3 of T

T row 3 is str inc

T row 3: $\quad 2\ 3\ 5 \quad \leftarrow 3$

3 bumped from row 2 of T

T row 2 str inc

T row 2 : $1\ 3\ 4\ 6 \leftarrow 3$

3 bumped from row 1 of T (by something ≤ 3)

T row 1 str inc

T row 1 : $1\ 3\ 4\ 6 \leftarrow 2$

T :

$$\begin{matrix} 1 & 3 & 4 & 6 \\ 1 & 3 & 4 & 6 \\ 2 & 3 & 5 & \\ 2 & 4 & & \\ 3 & & & \\ 5 & & & \\ 7 & & & \end{matrix}$$

$j = 2$

We now describe dual RSK via dual Schensted ins

ex $k=3$ $n=4$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Find corresp P, Q.

A in 2-line notation (as before)

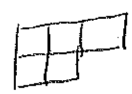
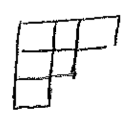
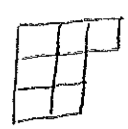
$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & 2 & 3 & 4 & 1 & 3 \end{bmatrix}$$

Get P from row 2

Apply

$$2 \Leftarrow 3 \Leftarrow 2 \Leftarrow 3 \Leftarrow 4 \Leftarrow 1 \Leftarrow 3$$

In stages

Stage	Insertion sequence	record shapes
1	2	
	2 3	
2	2 3 2	
	2 3 2 3	
	2 3 4 2 3	
3	1 3 4 2 3 2	
	1 3 4 $p^t =$ 2 3 = d. s. stand 2 3	

$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & & \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & \\ \cline{1-2} 3 & 3 & \\ \cline{1-2} \end{array}$$

By constr

P = s. stand tabl entries $\in \{1, 2, \ldots, n\}$

show Q = s. stand tabl $\underbrace{\text{entries} \in \{1, 2, \ldots, k\}}_{\text{ok.}}$

By constr

P, Q conjugate shp

LEM Given dual semi stand Tableau T

Given pos integers ..

$$i < j$$

Fn $T \Leftarrow i \Leftarrow j$

Consider

bumping path fr i $*$

bumping path fr j $**$

Then

(i) length $(*) \geq$ length $(**)$

(ii) Fn $1 \leq r \leq$ length $(**)$

Loc in row r of $*$ is strictly to left of
Loc in row r of $**$

pf Use ind on # rows in T $\square$

LEM Given dual s. stand tableau T shp λ

Given positive integers

$$a_1 < a_2 < \cdots < a_\ell$$

Consider insertions

$$T \Leftarrow a_1 \Leftarrow a_2 \Leftarrow \cdots \Leftarrow a_\ell$$

which yield dual s. stand tableau shp μ.

Then μ/λ is horiz strip. $\square$

pf. By prev lemma.

LEM Ref to dual RSK via dual

Schensted insertion

Suppose input A yields (P, Q).

then

Q = semi stand tableau with entries $\in \{1, 2, \dots, k\}$.

pf Use prev Lem. $\square$

<u>Caution</u>. For dual Schensted insertion

$$A \longrightarrow (P, Q)$$

does not imply

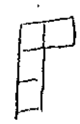
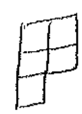
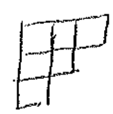
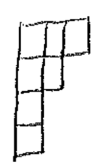
$$A^t \longrightarrow (Q, P)$$

ex, cont

$$A^t = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A^t in 2-line notation

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 3 & 1 & 2 & 1 & 2 & 3 & 2 \end{bmatrix}$$

stage	insertion sequence	record shapes
1	3	
2	1 3	
	1 2 3	
3	1 2 1 3	
	1 2 1 2 3	
	1 2 3 1 2 3	
4	1 2 3 1 2 2 3	

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$