Lecture 36 Monday Nov 25

Another example of extended RSK

ex $k=3$ $n=4$

$$A = \begin{bmatrix} 3 & 1 & 4 & 2 \\ 0 & 2 & 1 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Use extended RSK to find P, Q

A in 2-line notation:

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2&3&3&3&3&3&3&3&3&3&3 \\ 1&1&1&2&3&3&3&3&4&4&2&2&3&4&4&4&4&4&1&1&1&1&2&2&2&3&3&4 \end{bmatrix}$$

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stage	insert
1	1 1 1 2 3 3 3 3 4 4
2	1 1 1 2 2 3 3 3 4 4 3 1 1 1 2 2 2 3 3 4 4 3 3 1 1 1 2 2 2 3 3 3 4 3 3 4 1 1 1 2 2 2 3 3 3 4 4 4 4 4 4 3 3 4
3	1 1 1 1 2 2 3 3 3 4 4 4 4 4 4 2 3 4 3 1 1 1 1 1 2 3 3 3 4 4 4 4 4 4 2 2 4 3 3 1 1 1 1 1 1 3 3 3 4 4 4 4 4 4 2 2 2 3 3 4 1 1 1 1 1 1 1 3 3 4 4 4 4 4 4 2 2 2 3 3 3 4 1 1 1 1 1 1 1 2 3 4 4 4 4 4 4 2 2 2 3 3 3 3 4 1 1 1 1 1 1 1 2 2 4 4 4 4 4 4 2 2 2 3 3 3 3 3 4 1 1 1 1 1 1 1 2 2 2 4 4 4 4 4 2 2 2 3 3 3 4 3 3 4 1 1 1 1 1 1 1 2 2 2 3 4 4 4 4 2 2 2 3 3 3 4 4 3 3 4 1 1 1 1 1 1 1 2 2 2 3 3 4 4 4 2 2 2 3 3 3 4 4 4 3 3 4 1 1 1 1 1 1 1 2 2 2 3 3 4 4 4 4 2 2 2 3 3 3 4 4 4 3 3 4 = ρ

1 1 1 1 1 1 1 1 1 2 2 2 2 2 3
2 2 2 3 3 3 3 3
3 3 3

= φ

Repeat using

$$A^t = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

A^t in 2-line notation:

$$\begin{bmatrix} 1&1&1&1&1&1&1&2&2&2&2&2&2&3&3&3&3&3&3&3&4&4&4&4&4&4&4&4 \\ 1&1&1&3&3&3&3&1&2&2&3&3&3&1&1&1&1&2&3&3&1&1&2&2&2&2&2&3 \end{bmatrix}$$

<table>
<tr><th>stage</th><th>insert</th></tr>
<tr><td>1</td><td>1113333</td></tr>
<tr><td>2</td><td>1111333
3

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1111223333
333</td></tr>
<tr><td>3</td><td>1111123333
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111 111 1111 33
222 3333
333

111 111 1111 23
222 33333
333

111 111 1111 22
222 333 333
333

$\boxed{\begin{matrix}111\ 111\ 1111\ 22223 \\ 222\ 333\ 333 \\ 333\end{matrix}}$</td></tr>
</table>

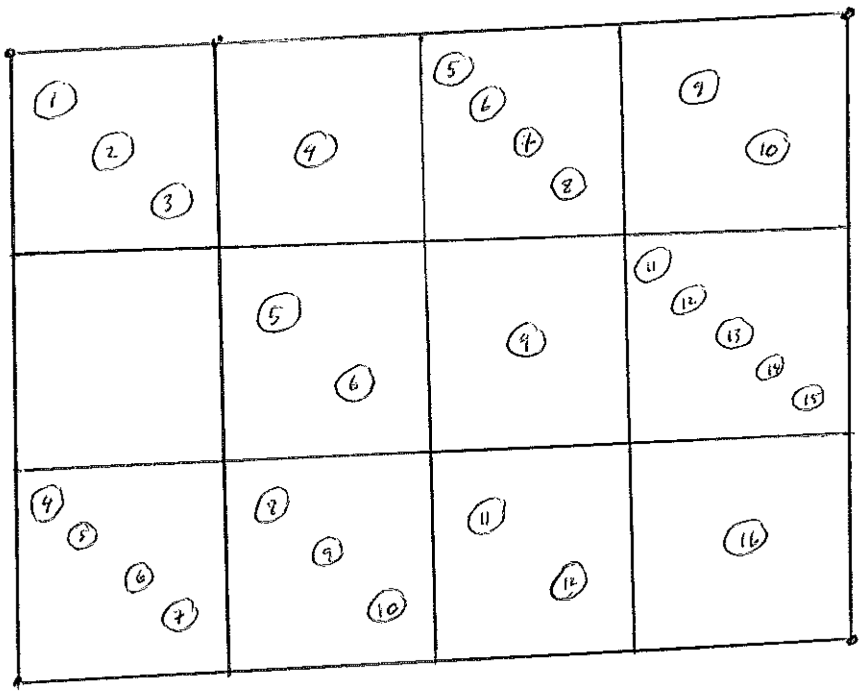
$= P^{new} = Q$

$\boxed{\begin{matrix}11111111\ 22233\ 4444 \\ 222333444 \\ 334\end{matrix}}$

$= Q^{new} = P$

Next, find P, Q using box/ball

$A^{(1)}$



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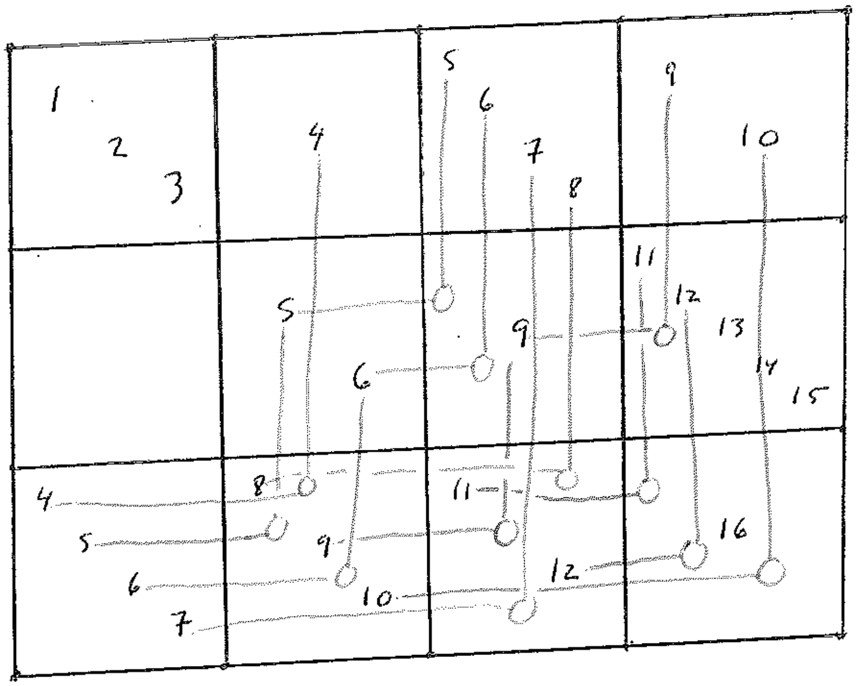
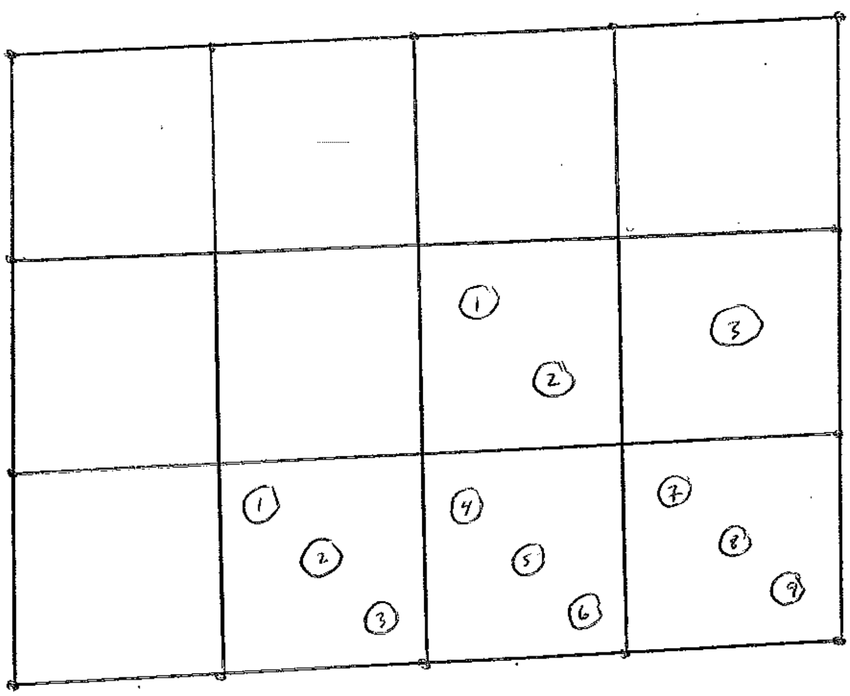
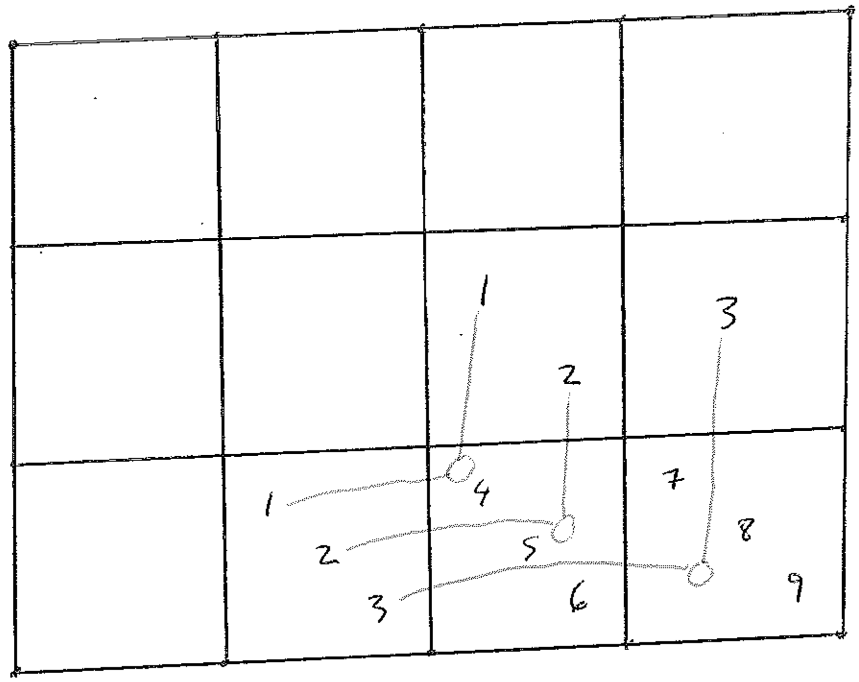
Find $A^{(2)}$



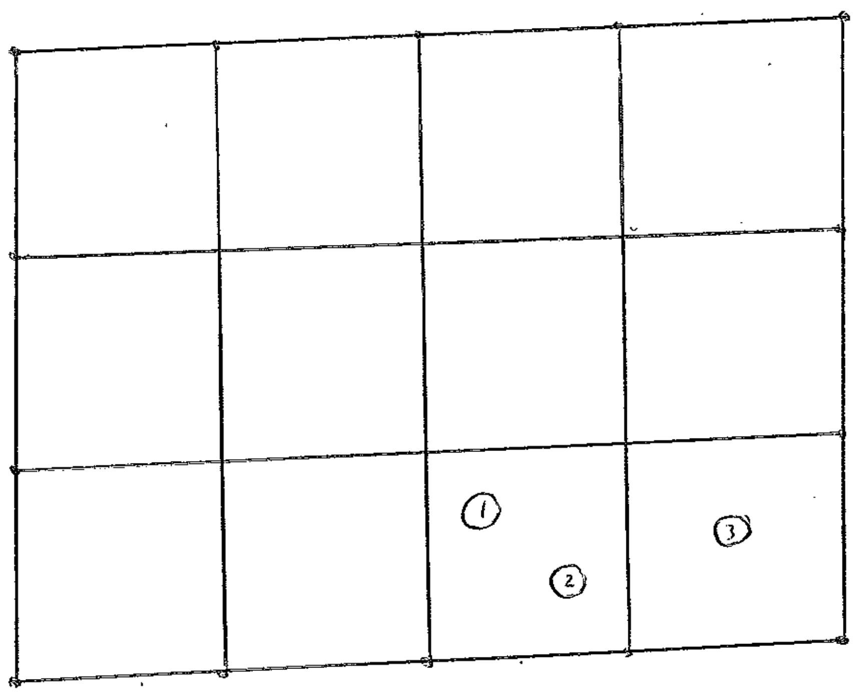
$A^{(2)}$



Find $A^{(3)}$



$A^{(3)}$



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Using $A^{(1)}, A^{(2)}, A^{(3)}$ find P, Q

P:

1 1 1 1 1 1 1 2 2 2 3 3 4 4 4 4
2 2 2 3 3 3 4 4 4
3 3 4

Q:

1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 3
2 2 2 3 3 3 3 3 3
3 3 3

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Dual RSK

This variation on RSK gives a bijection between

(i) the set of $k \times n$ matrices that have entries in $\{0,1\}$

(ii) the set of ordered pairs (P, Q) such that

P = s. stand tableau with entries in $\{1,2,\dots,n\}$,

Q = s. stand tableau with entries in $\{1,2,\dots,k\}$

P, Q have conjugate shape

Next goal: use crystals to motivate dual RSK

Fix $\Phi = A_r$ $\quad \Lambda = GL(r+1)$, $\quad n = r+1$

For $\ell \geq 1$ recall crystal $B_{(1^\ell)}$ has vertex set

$$\begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_\ell \\ \hline \end{array} \qquad 1 \leq a_1 < a_2 < \cdots < a_\ell \leq n$$

For $k \geq 1$ consider crystal of form

$$B_{(1^{\ell_k})} \otimes \cdots \otimes B_{(1^{\ell_2})} \otimes B_{(1^{\ell_1})} \qquad ☆$$

For
$$x = x_k \otimes \cdots \otimes x_2 \otimes x_1 \in ☆$$

represent x by $k \times n$ matrix:

For $1 \leq i \leq k$ and $1 \leq j \leq n$

$$(i,j)\text{-entry} = \#\text{ boxes in } x_i \text{ that contain } j$$
$$\in \{0, 1\}$$

For this matrix

$$i\text{th row sum} = \#\text{ boxes in } x_i = \ell_i$$
$$(1 \leq i \leq k)$$

ex $\quad n=4 \quad k=3$

Fn

$$X = X_3 \otimes X_2 \otimes X_1$$

$$X_1 = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array} \qquad l_1 = 2$$

$$X_2 = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline\end{array} \qquad l_2 = 3$$

$$X_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \qquad l_3 = 3$$

corresp matrix $\qquad\qquad$ row sum

$$\begin{array}{c c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{c} 2 \\ 3 \\ 3 \end{array} \end{array}$$

Above corresp gives bijection between

(i) $B_{(1^{l_k})} \otimes \cdots \otimes B_{(1^{l_2})} \otimes B_{(1^{l_1})}$

(ii) the set of $k \times n$ matrices with entries in $\{0,1\}$ and row sums $l_1, l_2, \ldots, l_k$.

Define crystal $\mathcal{B} = \mathcal{B}(k,n)$ by

$$\mathcal{B} = \bigcup_{l_1, l_2, \ldots, l_k \in \mathbb{N}} B_{(1^{l_k})} \otimes \cdots \otimes B_{(1^{l_2})} \otimes B_{(1^{l_1})}$$

Above bijection induces bijection between

(i) $\mathcal{B}(k,n)$

(ii) set of $k \times n$ matrices with entries in $\{0,1\}$

Next we find the hw elements in $B(k,n)$

Recall crystal

$$B_{(1^{\ell_k})} \otimes \cdots \otimes B_{(1^{\ell_2})} \otimes B_{(1^{\ell_1})} \qquad ☆$$

For a partition $\lambda \in \Lambda^+$ we give a bijection between

(i) set of hw elements $x \in ☆$ with wt λ

(ii) set of semi standard tableaux Q of shp λ^t with entries in $\{1, 2, \dots, k\}$ such that exactly

- ℓ_1 boxes of Q contain 1
- ℓ_2 ---- 2
- --- ---
- ℓ_k --- k

For $x \in ☆$ describe corresp Q

Write $x = x_k \otimes \cdots \otimes x_2 \otimes x_1$

For $1 \le i \le k$ the entries of X_i give the columns of Q that contain i

ex $k=3,\ n=7$

$$X = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 6 \\ \hline 7 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ 2 \\ 3 \\ \hline\end{array}$$

$$Q = \begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 & & & \\ 3 & & & & & & \end{array}$$

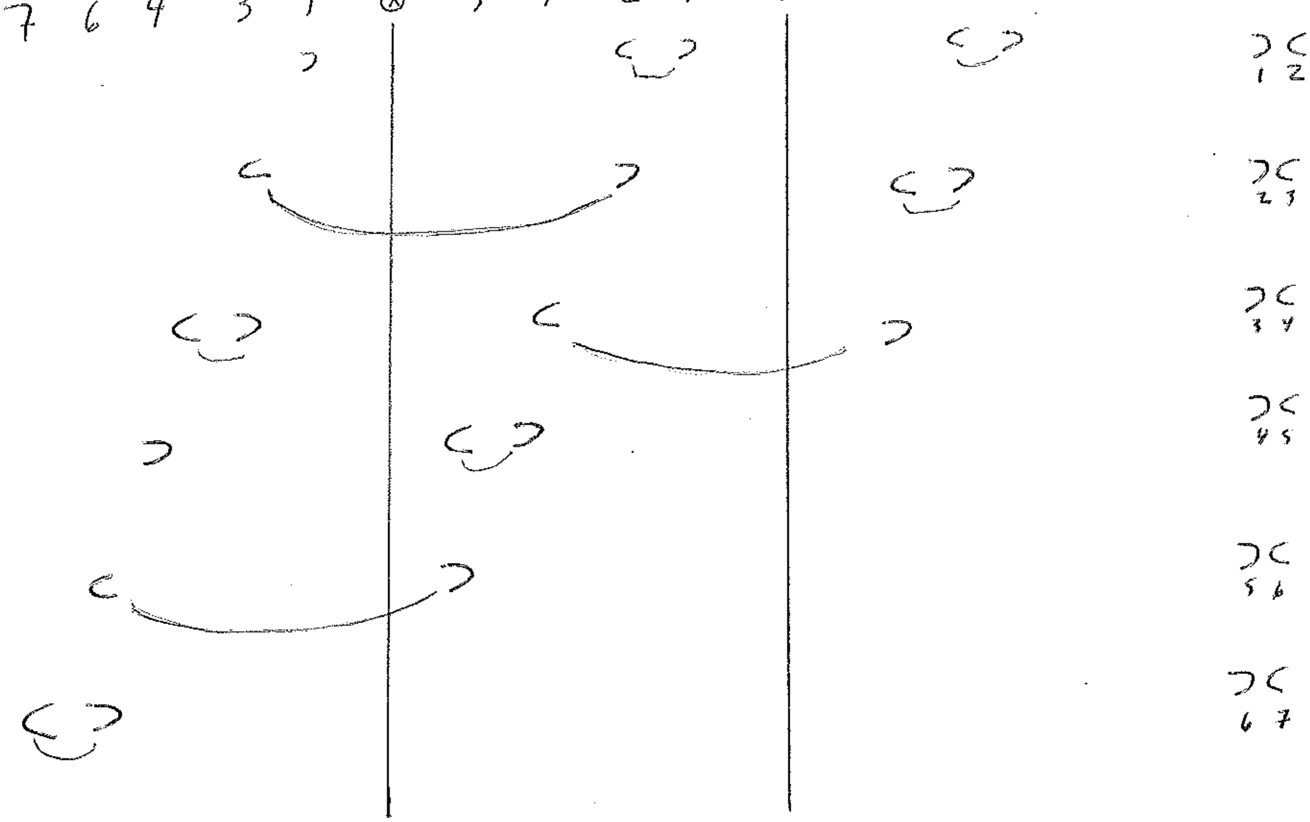
obs $(\text{shp}\ Q)^t = (3,2,2,2,1,1,1) = wt(x)$

check x " hw

RowR(x): $\cancel{7}\,6431 \otimes 5421 \otimes 321$

Apply signature rule

$$7 \quad 6 \quad 4 \quad 3 \quad 1 \quad \otimes \quad 5 \quad 4 \quad 2 \quad 1 \quad \otimes \quad 3 \quad 2 \quad 1$$



So $\quad \varepsilon_i(x) = 0 \quad 1 \le i \le 6$

x is hw ✓

For a partition $\lambda \in \Lambda^+$

above bijection induces a bijection between

(i) the set of hw elements in $\mathcal{B}(k,n)$ that have wt λ

(ii) the set of semi standard tableaux φ of shp λ^t that have entries in $\{1,2,\dots,k\}$

We are now ready to describe the dual RSK

Dual RSK gives a bijection between

(i) the set of $k \times n$ matrices that have entries in $\{0,1\}$

(ii) the set of ordered pairs (P, Q) such that

P = s. stand tableau with entries $\in \{1,2,\dots,n\}$

Q = s. stand tableau with entries $\in \{1,2,\dots,k\}$

P, Q have conjugate shape

Fix $x \in$ (i) we now describe corresp $(P, Q) \in$ (ii)

We may identify x with an element in $\mathcal{B}(k,n)$

$x \in$ connected component C of $\mathcal{B}(k,n)$

$\exists$ partition $\lambda \in \Lambda^+$ st

C iso B_λ

the crystal iso

$C \to B_\lambda$

sends $x \to P$

By constr

P = s_1 stand tableau shp λ

with entries $\in \{1, 2, \dots, n\}$

Also,

C has unique hw element (call it $\bar{x}$)

$\bar{x}$ has wt λ

So $\bar{x}$ corresponds to s_1 standard tableau Q

of shp λ^t that has entries $\in \{1, 2, \dots, k\}$

Note P, Q have conjugate shape.

We have obtained (P, Q)