Lecture 35 Friday Nov 22

LEM Given s. stand tableau $T$

Given $i, j \in \mathbb{N}$ s.t

$i \leq j$

For $T \leftarrow i \leftarrow j$

consider

bumping path for $i$ $\quad (*)$

bumping path for $j$ $\quad (**)$

then

(i) length $(*) \geq$ length $(**)$

(ii) For $1 \leq r \leq$ length$(**)$,

Loc in row $r$ of $*$ is strictly to left of

Loc in row $r$ of $**$

pf Use induction on # rows for $T$ $\square$

ex $T =$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 4 & 5 \\ \cline{1-3} 5 & 5 \\ \cline{1-2} \end{array}$$

$i = 1,$

$j = 1$

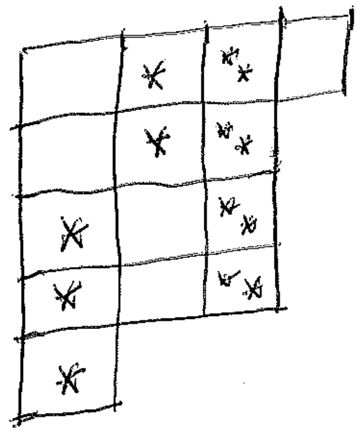
$T \leftarrow 1 =$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

$T \leftarrow 1 \leftarrow 1 =$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 4 & 4 \\ \cline{1-3} 4 & 5 & 5 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$$

Bumping paths for i and j



LEM Given s. stand tableau $T$ shp $\lambda$

Given natural numbers

$a_1 \le a_2 \le \cdots \le a_\ell$

Consider insertions

$T \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_\ell$

which yield s. stand tableau shp $\mu$.

then $\mu/\lambda$ is horizontal strip

pf By prev Lemma. $\square$

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LEM Ref to extended RSK via Schensted insertion,

Suppose a given input matrix $A$ yields the output $(P, Q)$.

then

$Q$ = $S_i$ standard tableau with entries $\in \{1, 2, \ldots, k\}$

pf By constr

$Q$ has all entries $\in \{1, 2 \ldots k\}$

By constr.

For each row of $Q$ the entries are weakly inc $\rightarrow$.

Show for each col of $Q$ the entries are strictly inc $\downarrow$

For $1 \leq i \leq k$ the boxes in $Q$ that contain $i$ are the boxes added to $P$ during stage $i$

these boxes form horiz strip by prev lem.

Result follows. $\square$

Ref to extended RSK via Schensted Insertion,
we now reverse the direction.

Start with $(P, Q)$ s.t

$P =$ s. stand tableau with entries in $\{1,2,\dots,n\}$

$Q =$ s. stand tableau with entries in $\{1,2,\dots,k\}$

$P, Q$ have same shape.

Find the corresp $k \times n$ matrix $A$ that has entries in $\mathbb{N}$.

Suffices to express $A$ in 2-line notation $\ast$

ex $n=4$ $k=3$

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

From entries in $Q$ we find top row of $\ast$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ & & & & & & & & & \end{bmatrix}$$

The second row of $\ast$ is the sequence that yields the above $P$ and the standard tableau

$$Q' = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 & 7 & 10 \\ \hline 5 & 9 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

via classical RSK Schensted insertion.

| Given tableau with new location | find "previous" tableau and inserted element |
|---|---|
| 1 1 2 2 2 3 $\boxed{3}$<br>2 3<br>4 | 1 1 2 2 2 3<br>2 3<br>4 $\leftarrow$ 3 |
| 1 1 2 2 2 3<br>2 $\boxed{3}$<br>4 | 1 1 2 2 3 3<br>2<br>4 $\leftarrow$ 2 |
| 1 1 2 2 3 3<br>2<br>$\boxed{4}$ | 1 2 2 2 3 3<br>4 $\leftarrow$ 1 |
| 1 2 2 2 3 $\boxed{3}$<br>4 | 1 2 2 2 3<br>4 $\leftarrow$ 3 |
| 1 2 2 2 $\boxed{3}$<br>4 | 1 2 2 2<br>4 $\leftarrow$ 3 |
| 1 2 2 2<br>$\boxed{4}$ | 1 2 2 4 $\leftarrow$ 2 |
| 1 2 2 $\boxed{4}$ | 1 2 2 $\leftarrow$ 4 |
| 1 2 $\boxed{2}$ | 1 2 $\leftarrow$ 2 |
| 1 $\boxed{2}$ | 1 $\leftarrow$ 2 |
| $\boxed{1}$ | 1 |

* becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 4 & 2 & 3 & 3 & 1 & 2 & 3 \end{bmatrix}$$

Ref to extended RSK via Schensted insertion,

we have seen the example

$$A \longleftrightarrow (P, Q)$$

for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$P = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$Q = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

We now replace $A$ by its transpose $A^t$

$$A^t : \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$A^t$ in 2-line notation

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 1 & 3 & 1 & 1 & 2 & 3 & 2 & 2 & 3 & 1 \end{bmatrix}$$

Next, apply Schensted insertion to 2nd row,

in stages

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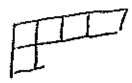
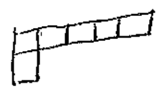
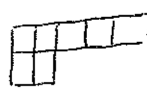
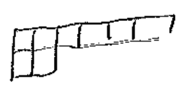
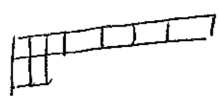
| Stage | Insertion sequence | record shps |
|---|---|---|
| 1 | 1 |  |
| | 13 |  |
| 2 | 1 1<br>3 |  |
| | 1 1 1<br>3 |  |
| | 1 1 1 2<br>3 |  |
| | 1 1 1 2 3<br>3 |  |
| 3 | 1 1 1 2 2<br>3 3 |  |
| | 1 1 1 2 2 2<br>3 3 |  |
| | 1 1 1 2 2 2 3<br>3 3 |  |
| 4 | $\begin{array}{\|c\|c\|c\|c\|c\|c\|c\|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$ | $\begin{array}{\|c\|c\|c\|c\|c\|c\|c\|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ |
| | $= P^{new} = Q$ | $= Q^{new} = P$ |

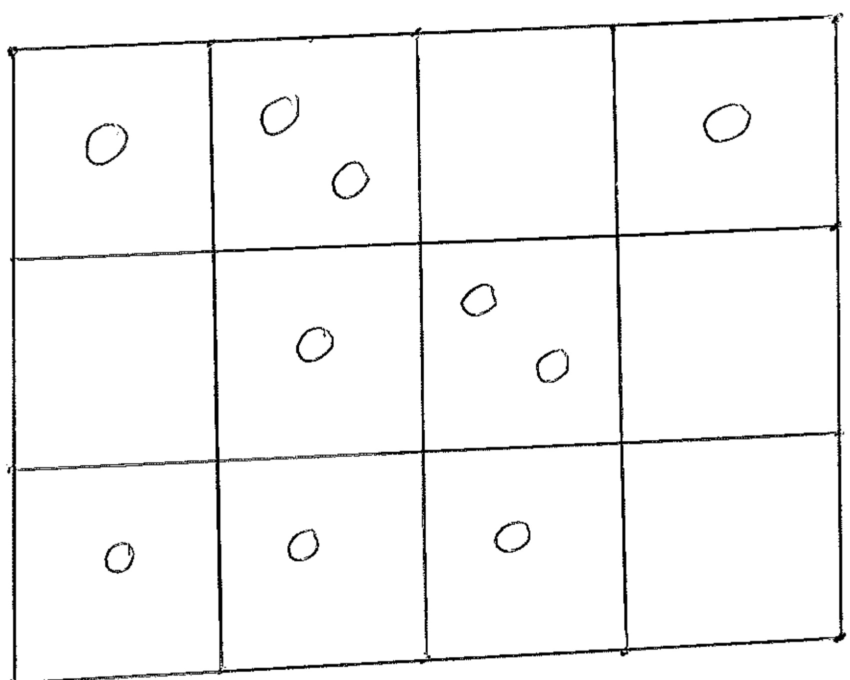
Replacing $A$ by $A^t$ causes $P, Q$ to swap!

To illustrate why $A \leftrightarrow A^t$ sends $P \leftrightarrow Q$,
we give the "box/ball" interp of extended RSK
via Schensted insertion.

ex $\quad k=3 \quad n=4$

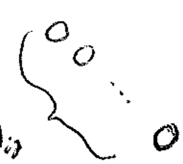
$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Consider a $k \times n$ rectangle containing balls:



In row $i$ and col $j$ put $A_{ij}$ balls in order

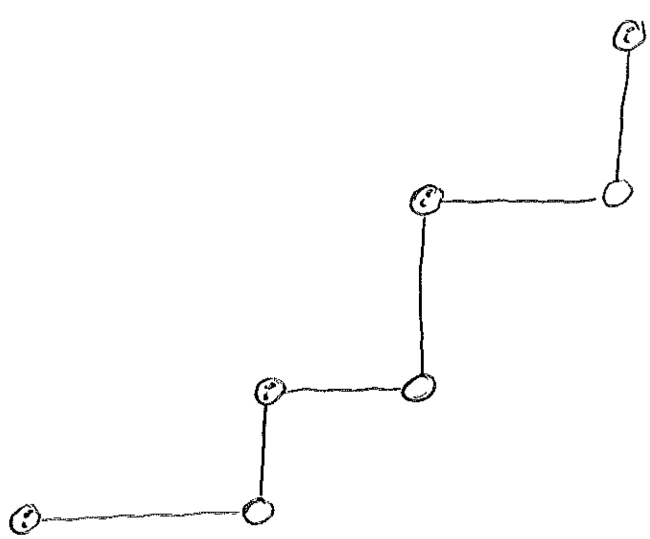
$A_{ij}$

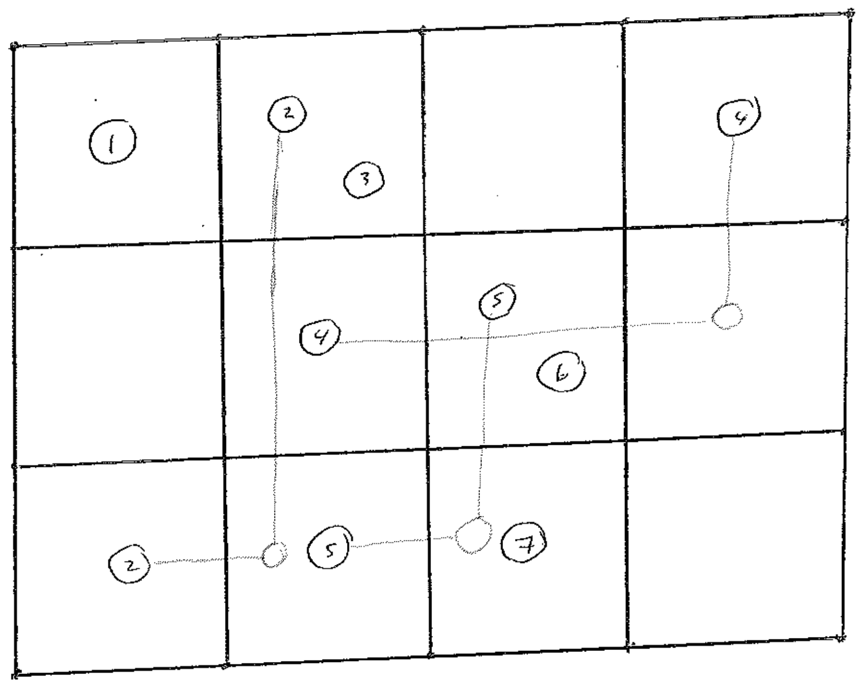


Fill each ball with the min pos integer such that each row and col is strictly inc'g

| | | | |
|---|---|---|---|
| 1 | 2 3 | | 4 |
| | 4 | 5 6 | |
| 2 | 5 | 7 | |

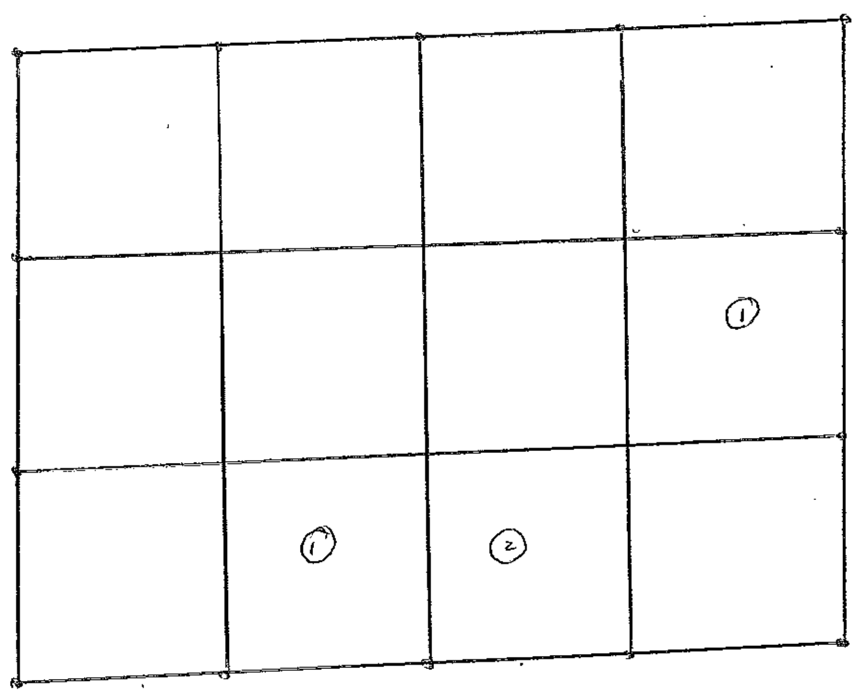
Next, for each type of ball create a staircase path:





Use created balls to make $A^{(2)}$:

$A^{(2)}$



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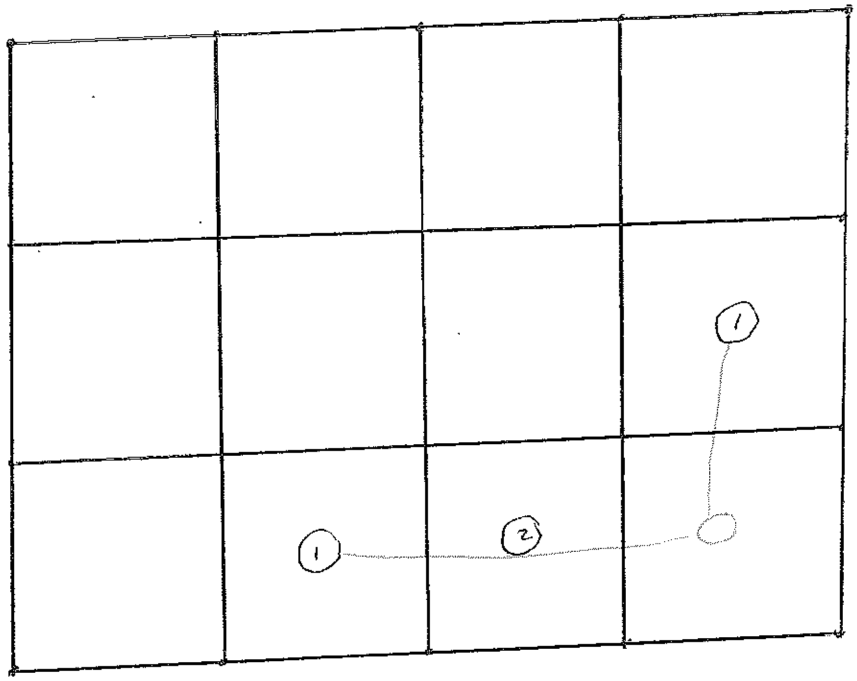
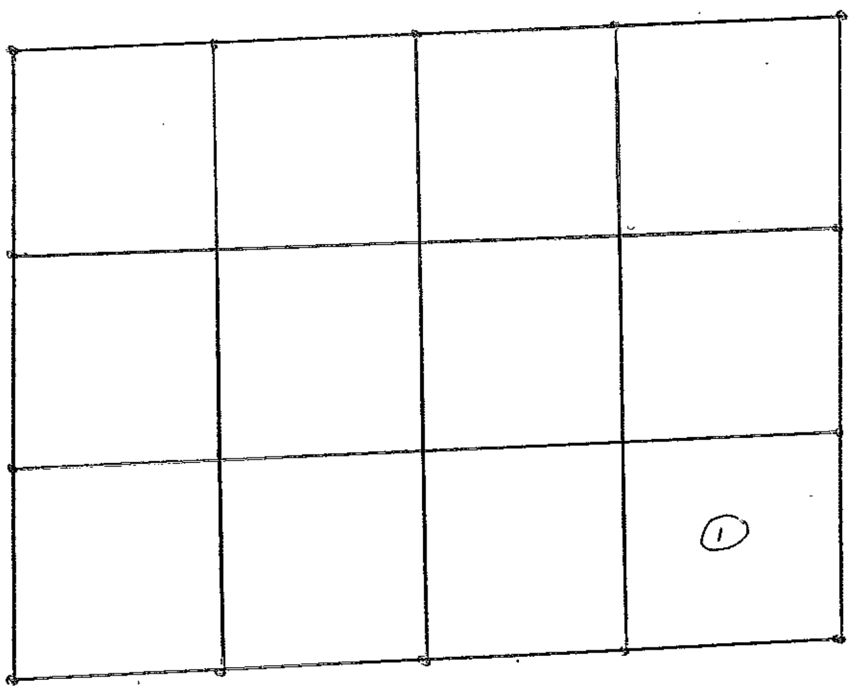
Iterate to get $A^{(3)}$




$A^{(3)}$



$A^{(1)}$ describes row 1 of $P$, $Q$

$A^{(2)}$ ... 2 --

$A^{(3)}$ -- 3 ---

## $A^{(1)}$

For $1 \le j \le n$

\# "new" elements in col $j$ of $A^{(1)}$ = # $j$'s in row 1 of $P$

$P$ row 1: 1 1 2 2 2 3 3

For $1 \le i \le k$

\# "new" elements in row $i$ of $A^{(1)}$ = # $i$'s in row 1 of $Q$

$Q$ row 1: 1 1 1 1 2 2 3

## $A^{(2)}$

For $1 \le j \le n$

\# "new" elements in col $j$ of $A^{(2)}$ = # $j$'s in row 2 of $P$

$P$ row 2: 2 3

For $1 \le i \le k$

\# "new" elements in row $i$ of $A^{(2)}$ = \# $i$'s in row 2 of $Q$

$Q$ row 2:

2 3

## $A^{(3)}$

For $1 \le j \le n$

\# "new" elements in col $j$ of $A^{(3)}$ = \# $j$'s in row 3 of $P$

$P$ row 3: 4

For $1 \le i \le k$

\# "new" elements in row $i$ of $A^{(3)}$ = \# $i$'s in row 3 of $Q$

$Q$ row 3: 3

So

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$