Lecture 34 Wed Nov 20 11/20/19

Above corresp gives bijection between

(i) $B_{(l_k)} \otimes \cdots \otimes B_{(l_2)} \otimes B_{(l_1)}$

(ii) the set of $k \times n$ matrices with entries in $\mathbb{N}$ and row sums $l_1, l_2, \ldots, l_k$

Define crystal $\mathcal{B} = \mathcal{B}(k,n)$ by

$$\mathcal{B} = \bigcup_{l_1, l_2, \ldots, l_k \in \mathbb{N}} B_{(l_k)} \otimes \cdots \otimes B_{(l_2)} \otimes B_{(l_1)}$$

Above bijection induces a bijection between

(i) $\mathcal{B}(k,n)$

(ii) set of $k \times n$-matrices with entries in $\mathbb{N}$

Next we find the hw elements in $\mathcal{B}(k,n)$.

Recall the crystal

$$B_{(\ell_k)} \otimes \cdots \otimes B_{(\ell_2)} \otimes B_{(\ell_1)} \qquad \text{☆}$$

For a partition $\lambda \in \Lambda^+$ we give a bijection between

(i) set of hw elements $x \in$ ☆ with wt λ

(ii) set of semi standard tableaux φ of shape λ with entries in $\{1, 2, \ldots, k\}$, such that exactly

- ℓ_1 boxes of φ contain 1
- ℓ_2 ... 2
- ... ... k
- ℓ_k ...

For $x \in$ ☆ describe corresponding φ

Write $\quad x = x_k \otimes \cdots \otimes x_2 \otimes x_1$

For $1 \le i \le k$ the entries in X_i describe the Locations in Q that contain i.

For $1 \le j \le n$,

$\#$ boxes in X_i that contain j

$=$ $\#$ boxes in row j of Q that contain i

ex $\quad n=3 \quad k=4$

$$X = \boxed{2 \,|\, 2 \,|\, 3} \otimes \boxed{1 \,|\, 1 \,|\, 2 \,|\, 3} \otimes \boxed{1 \,|\, 2} \otimes \boxed{1 \,|\, 1}$$

$$\varphi = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & \\ \cline{1-4} 3 & 4 & & & \\ \cline{1-2}\end{array}$$

obs

shp of $\varphi = (5,4,2) = wt(x)$

check X is hw

Apply signature rule $\quad \begin{array}{cc} 1 & 2 \\ > & < \end{array}$

$$\begin{array}{ccccccccccccc} 2 & 2 & 3 & \otimes & 1 & 1 & 2 & 3 & \otimes & 1 & 2 & \otimes & 1 \;\; 1 \\ < & < & & & > & > & < & & & > & < & & > \;\; > \end{array}$$

reduces to $\quad >$

$\varphi_1(x) = 1, \quad \varepsilon_1(x) = 0$
$\quad\quad\quad\quad\quad\quad\quad\quad \uparrow$

Apply sign rule $\begin{matrix} 2 & 3 \\ > & < \end{matrix}$

$$\begin{matrix} 2 & 2 & 3 & \otimes & 1 & 1 & 2 & 3 & \otimes & 1 & 2 & \otimes & 1 & 1 \\ > & > & < & & & & > & < & & & > & & & \end{matrix}$$

reduces to $>>$

$\varphi_2(x) = 2,$ $\quad\varepsilon_2(x) = 0$

$\uparrow$

X is hw

For a partition $\lambda \in \Lambda^+$, above bijection induces a bijection between

(i) the set of hw elements $x \in \mathcal{B}(k,n)$ that have wt λ

(ii) the set of semi standard tableaux Φ of shp λ that have entries in $\{1, 2, \dots, k\}$

We are now ready to describe the extended RSK.

Extended RSK gives a bijection between

(i) The set of $k \times n$ matrices that have entries in $\mathbb{N}$

(ii) The set of ordered pairs (P, Q) such that

P = s. stand tableau with entries in $\{1, 2, \ldots, n\}$

Q = s. stand tableau with entries in $\{1, 2, \ldots, k\}$

P, Q have same shape.

For $x \in$ (i) we now describe corresp $(P, Q) \in$ (ii)

We may identify x with an element in $\mathcal{B}(k, n)$

$x \in$ connected component C of $\mathcal{B}(k, n)$

$\exists$ partition $\lambda \in \Lambda^+$ st

C iso B_λ

The crystal iso

$$C \to B_\lambda$$

sends $\quad x \to P$

By construction,

$P =$ s. stand tableau of shp λ with entries in $\{1, 2, \ldots, n\}$

Also,

C has unique hw element (call it $\bar{x}$)

$\bar{x}$ has wt λ

So $\bar{x}$ corresponds to a s. standard tableau Q of shp λ with entries in $\{1, 2, \ldots, k\}$.

P, Q have same shp λ.

We have obtained (P, Q)

Next goal: describe extended RSK using Schensted insertion

Start with $k \times n$-matrix A with entries in $\mathbb{N}$

ex $k=3$ $n=4$

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad *$$

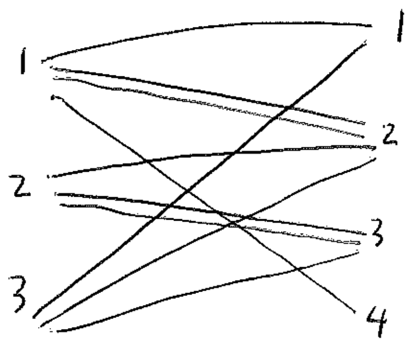
Consider a bipartite graph with vertices

$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ k & n \end{array}$$

For $1 \le i \le k$ and $1 \le j \le n$ the graph has

A_{ij} edges between i and j

For $*$



$**$

For above graph, represent each edge

i ——— j

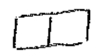
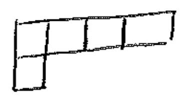
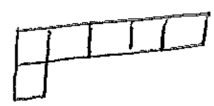
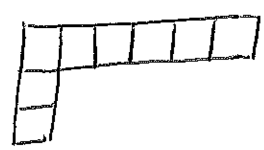
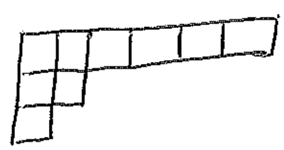
by column vector

$$\begin{bmatrix} i \\ j \end{bmatrix}$$

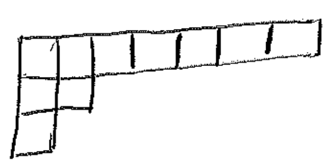
List the resulting column vectors in LEX order (viewing $\begin{bmatrix} i \\ j \end{bmatrix}$ as a word ij) to get A in 2-line notation. * yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 4 & 2 & 3 & 3 & 1 & 2 & 3 \end{bmatrix}$$

Next apply Schensted Insertion to 2nd row, in "stages" suggested by 1st row

Stage	Insertion sequence	record shapes
1	1	
	12	
	122	
	1224	
2	1222 4	
	12223 4	
	122233 4	
3	112233 2 4	
	112223 23 4	

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&2&2&3&3\\\hline 2&3\\\cline{1-2} 4\\\cline{1-1}\end{array} = P$$



describe sequence of shapes in recording tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array} = Q$$

By constr

$P =$ s. stand tableau with entries in $\{1, 2, \dots, n\}$

Show

$Q =$ s. stand tableau with $\underbrace{\text{entries in } \{1, 2, \dots k\}}_{\text{this part OK}}$

By constr

P, Q same shp