Lecture 33 Monday Nov 18 11/18/19

For a semi standard tableau T and

$1 \le i \le n$

Suppose we are given

$T \leftarrow i$ and new location.

Recover T and i

ex

$$T \leftarrow i: \quad \begin{array}{ccccccc} 1 & 1 & 2 & 2 & 3 & 5 & 7 \\ 2 & 3 & 3 & 3 & 4 & & \\ 4 & 4 & 5 & & & & \\ 6 & \boxed{6} & & & & & \end{array}$$

$\square$ = new Loc

6 bumped from row 3 of T

T row 3 is w inc

T row 3: $446 \leftarrow 5$

5 bumped from row 2 of T

T row 2 is w inc

T row 2: $23335 \leftarrow 4$

4 bumped from row 1 of T (by something smaller)

T row 1 is w inc

T row 1: 1 1 2 2 4 5 7 ← 3

T:

$$\begin{array}{lllllll} 1 & 1 & 2 & 2 & 4 & 5 & 7 \\ 2 & 3 & 3 & 3 & 5 & & \\ 4 & 4 & 6 & & & & \\ 6 & & & & & & \end{array}$$

$i = 3$

ex

$T \leftarrow i$:

$$\begin{array}{llllllll} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 & 5 \\ 4 & 4 & 5 & 5 & 6 \\ 5 & \boxed{5} \end{array}$$

$\square$ = new loc

Recover T and i

5 is bumped from T row 3. by something smaller

T row 3 is w inc

T row 3 : $4\ 5\ 5\ 5\ 6 \quad \leftarrow 4$

4 is bumped from T row 2 by something smaller

T row 2 is w inc

T row 2 : $3\ 4\ 4\ 4\ 5 \quad \leftarrow 3$

3 is bumped from T row 1 by something smaller

T row 1 is w inc

T row 1 : $1\ 1\ 2\ 3\ 3\ 3\ 4\ 4 \quad \leftarrow 2$

T :

$$\begin{array}{llllllll} 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 \\ 3 & 4 & 4 & 4 & 5 \\ 4 & 5 & 5 & 5 & 6 \\ 5 \end{array}$$

$i = 2$

Next we describe the RSK corresp

<u>Motivation</u> $\Phi = A_r$ $\quad \Lambda = GL(r+1)$, $\quad n = r+1$

$\mathbb{B}$ = standard crystal

Given $k \geq 1$. In Lec 16 we saw bij

$$\mathbb{B}^{\otimes k} \simeq \left\{ (P, Q) \;\middle|\; \begin{array}{l} P = \text{s. stand tableau entries in } \{1, 2, \ldots, n\} \\ Q = \text{stand. tableau} \\ P, Q \text{ same shp } \lambda \vdash k \end{array} \right\}$$

claimed to be
RSK corresp

In its classical form, RSK gives bijection between

(i) set of sequences $a_1 a_2 \ldots a_k$ such that

$a_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq k$

(ii) set of ordered pairs (P, Q) st

P = s. stand tableau entries in $\{1, 2, \ldots, n\}$

Q = stand tableau

P, Q same shp $\lambda \vdash k$

We desc RSK (i) $\to$ (ii) with an example

$n = 3$ $\qquad$ $k = 9$

$a_1 \cdots a_k =$ 2 1 3 1 1 3 2 1 2

Repeatedly insert $\quad a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_k$

record shps

2

$\begin{matrix} 1 \\ 2 \end{matrix}$

$\begin{matrix} 1 & 3 \\ 2 & \end{matrix}$

$\begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix}$

$\begin{matrix} 1 & 1 & 1 \\ 2 & 3 & \end{matrix}$

$\begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 3 & & \end{matrix}$

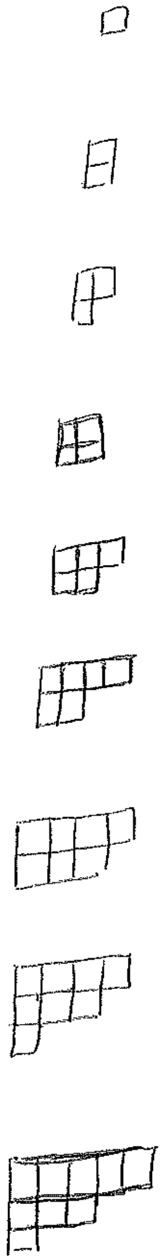
$\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \end{matrix}$

$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & \\ 3 & & & \end{matrix}$

$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$

$\parallel$

P



We describe the sequence of shapes in the following "recording tableau"

$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 9 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}$

$\parallel$

Q

By constr P, Q satisfy (ii)

This gives function (i) $\to$ (ii)

Given P, Q satisfying (ii) we can recover $q_1 q_2 \dots q_k$ by repeated application of recovery process

So our function (i) $\to$ (ii) is bijection

Ex $n = 3$ $k = 9$

$$P = \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{matrix} \qquad Q = \begin{matrix} 1 & 4 & 5 & 7 \\ 2 & 6 & 8 & 9 \\ 3 & & & \end{matrix}$$

Find $q_1 q_2 \cdots q_k$

Solution!

Given tableau with new Loc	Find "previous" tableau and inserted element
1 1 1 2 2 2 3 $\boxed{3}$ 3	1 1 1 3 2 2 3 3 $\leftarrow$ 2
1 1 1 3 2 2 $\boxed{3}$ 3	1 1 3 3 2 2 3 $\leftarrow$ 1
1 1 3 $\boxed{3}$ 2 2 3	1 1 3 2 2 3 $\leftarrow$ 3
1 1 3 2 $\boxed{2}$ 3	1 2 3 2 3 $\leftarrow$ 1
1 2 $\boxed{3}$ 2 3	1 2 2 3 $\leftarrow$ 3
1 $\boxed{2}$ 2 3	1 2 3 $\leftarrow$ 2
1 2 $\boxed{3}$	2 3 $\leftarrow$ 1
2 $\boxed{3}$	3 $\leftarrow$ 2
$\boxed{3}$	3

$a_1 a_2 \ldots a_9 =$ 3 2 1 2 3 1 3 1 2

Restricted RSK

Recall RSK

$a_1 a_2 \cdots a_k \implies (P, Q)$

$a_i \in \{1, 2, \ldots, n\}$

Assume $k = n$ and restrict RSK to those sequences $a_1 a_2 \cdots a_n$ such that $a_1, a_2, \ldots, a_n$ are mutually distinct

In this case, semi standard tableau P becomes standard

Restricted RSK gives bijection between

(i) permutations $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$

(ii) set of ordered pairs (P, Q) of standard tableaux that have same shape $\lambda \vdash n$

"Robinson - Schensted"

Extended RSK

Recall orig RSK:

$$a_1 a_2 \; a_k \quad \Longrightarrow \quad (\underset{\substack{\uparrow \\ s,\, st.}}{P}, \; \underset{\substack{\uparrow \\ st}}{Q})$$

$a_i \in \{1,2,\dots,n\}$

It is convenient to relabel:

$$a_k \cdots a_2 a_1 \quad \Longrightarrow \quad (\underset{\substack{\uparrow \\ s\ st.}}{P}, \; \underset{\substack{\uparrow \\ st.}}{Q})$$

$a_i \in \{1,2,\dots,n\}$

Represent each sequence $a_k \cdots a_2 a_1$ by a $k \times n$-matrix:
For $1 \le i \le k$, the matrix has entry 1 in row i and column a_i;
all other entries 0.

Ex $n=4$ $k=3$
the sequence $a_3 a_2 a_1 = 143$
is represented by the matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

We now extend RSK by replacing above $k \times n$ matrices by arbitrary $k \times n$-matrices with entries in $\mathbb{N} = \{0,1,2,\dots\}$

Extended RSK gives bijection between

(i): the set of $k \times n$-matrices that have entries in $\mathbb{N}$

(ii): the set of ordered pairs (P, Q) such that

P = s. stand tableau with entries in $\{1, 2, \dots, n\}$

Q = s. stand tableau with entries in $\{1, 2, \dots, k\}$

P, Q have same shape

Next goal: use crystals to motivate extended RSK.

For $\Phi = A_r$ $\quad \Lambda = GL(r+1)$, $\quad n = r+1$

For $\ell \geq 1$ recall the crystal $B(\ell)$ has vertex set

$$\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_\ell} \qquad a_i \in \{1, 2, \ldots, n\}$$

$$a_1 \leq a_2 \leq \cdots \leq a_\ell$$

For $k \geq 1$ consider crystal of form

$$B(\ell_k) \otimes \cdots \otimes B(\ell_2) \otimes B(\ell_1) \qquad \star$$

For

$$x = x_k \otimes \cdots \otimes x_2 \otimes x_1 \in \star$$

represent x by $k \times n$-matrix:

For $1 \leq i \leq k$ and $1 \leq j \leq n$

(i,j)-entry $=$ # boxes of x_i that contain j

For this matrix

ith row sum $=$ # boxes for x_i $= \ell_i$

$(1 \leq i \leq k)$

ex $n=4$ $k=3$

For

$$X = X_3 \otimes X_2 \otimes X_1$$

$X_1 = \boxed{1\ 1\ 1\ 2\ 2\ 3\ 4\ 4}$ $\ell_1 = 8$

$X_2 = \boxed{1\ 2\ 2\ 2\ 3\ 3}$ $\ell_2 = 6$

$X_3 = \boxed{2\ 2\ 2\ 2\ 2\ 3\ 4}$ $\ell_3 = 7$

corresponding matrix

row sums

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 2 & 1 & 2 \\ 1 & 3 & 2 & 0 \\ 0 & 5 & 1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} 8 \\ 6 \\ 7 \end{array}$$