Lecture 32 Friday Nov 15

Back to $\Phi = C_r$

wt lattice Λ from classification

$$\Lambda = \left\{ \sum_{i=1}^{r} a_i e_i \;/\; a_i \in \mathbb{Z} \right\}$$

For $\lambda \in \Lambda$

write $\lambda = \sum_{i=1}^{r} \lambda_i e_i$

Recall

$\lambda \in \Lambda^+$ iff $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$

(partition)

Assume $\lambda \in \Lambda^+$

DEF. With above notation, a

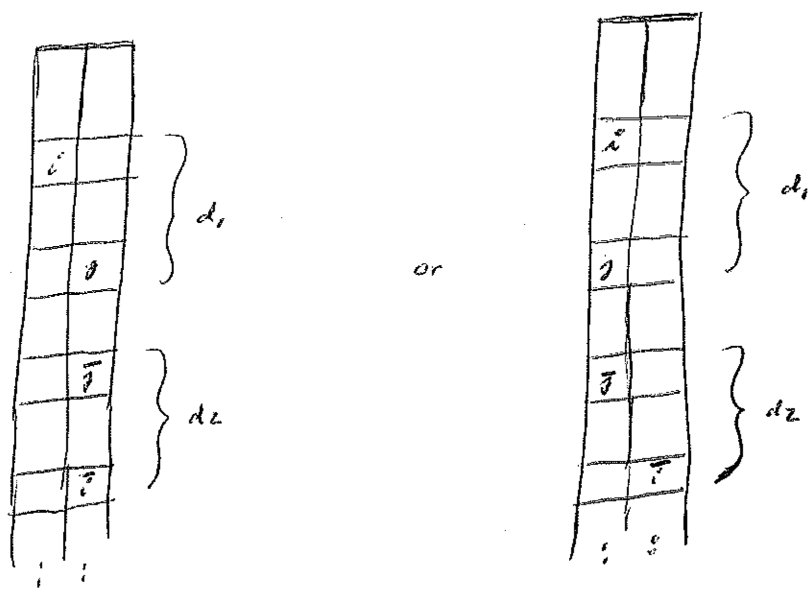
<u>C_r-tableau of shape λ</u>

is a filling T of the Ferrar diagram λ with entries from the C_r-alphabet such that:

C1: each column is good and balanced

C2: each row is weakly increasing

C3: If T has adjacent columns of form



$i \leq j$ $\qquad d_1 \geq 0, \qquad d_2 \geq 0$

then

$$d_1 + d_2 < j - i$$

Let

Tab_λ = set of C_r - crystals of shape λ

Thm For the map

$$ColR : Tab_\lambda \to \mathcal{B}^{\otimes|\lambda|}$$

the image is a connected component that is isomorphic to $\mathcal{B}_\lambda$

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The B_r - tableaux and D_r - tableaux are similarly defined, but the details are more complicated. See the text, Sections 6.3.2, 6.3.3.

Ch 7 Insertion algorithms

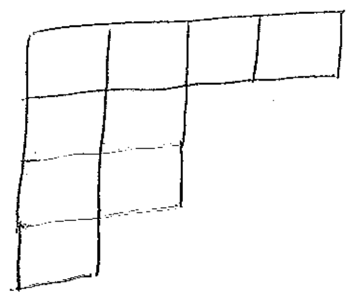
Our next focus is crystals of type A

Notation Given a partition λ, say

$$\lambda = (4, 2, 2, 1)$$

Recall Young diagram

$YD(\lambda)$:



Consider semistandard tableau T shp λ

$$T: \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \ast$$

By a **Location** in T we mean a box in $YD(\lambda)$

If the box is in row i, col j, then the box entry is denoted $T(i,j)$

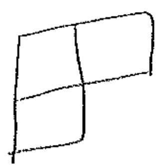
So far $*$,

$$T(2,1) = 2, \qquad T(1,3) = 1, \qquad T(3,2) = 4.$$

Given a second partition μ, say

$$\mu = (2,1)$$

$YD(\mu) =$



Call the ordered pair (λ, μ) a <u>skew shape</u> whenever

$$YD(\mu) \subseteq YD(\lambda)$$

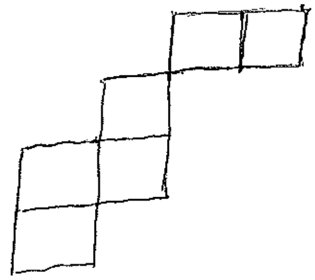
This skew shape is denoted λ/μ

the skew shape diagram $YD(\lambda/\mu)$

is obtained from $YD(\lambda)$ by removing $YD(\mu)$

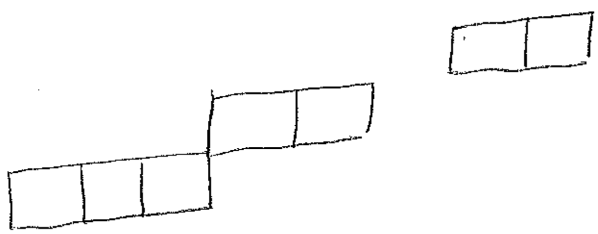
Ex

$YD(\lambda/\mu)$

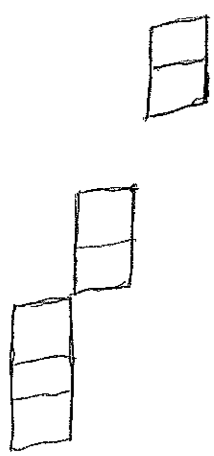


Def A skew shp λ/μ called a <u>horizontal strip</u> whenever no two boxes of λ/μ are in same column

For example



λ/μ is called a <u>vertical strip</u> whenever no two boxes of λ/μ are in same row. For example



Schensted insertion

Until further notice fix $n \geq 1$

Given semi standard tableau T with entries in alphabet $\{1, 2, \dots, n\}$

For $1 \leq i \leq n$, the Schensted insertion algorithm gives a way to insert an extra box $\boxed{i}$ into T, to get another semi standard tableau, denoted

$$T \leftarrow i$$

Case T has one row

ex $n=5$, T: 2 2 2 3 5 5

$T \leftarrow 1$:	1 2 2 3 5 5 2	"1 bumps 2"
$T \leftarrow 2$:	2 2 2 2 5 5 3	"2 bumps 3"
$T \leftarrow 3$:	2 2 2 3 3 5 5	"3 bumps 5"
$T \leftarrow 4$:	2 2 2 3 4 5 5	"4 bumps 5"
$T \leftarrow 5$:	2 2 2 3 5 5 5	

In summary, for

$T:\quad a_1\, a_2 \cdots a_t \qquad\qquad a_1 \le a_2 \le \cdots \le a_t$

Case	$T \leftarrow i$	Check $T \leftarrow i$ is semi standard
$i < a_1$	$\begin{matrix} i & a_2 \cdots a_t \\ a_1 & \end{matrix}$	$i < a_1$ OK
$\exists\, s\ (2 \le s \le t)$ such that $a_{s-1} \le i < a_s$	$\begin{matrix} a_1 \cdots a_{s-1} & i & a_{s+1} \cdots a_t \\ a_s & & \end{matrix}$	$a_1 \le a_{s-1} \le i < a_s$ so $a_1 < a_s$ OK
$a_t \le i$	$a_1\, a_2 \cdots a_t\, i$	$a_t \le i$, ok

<u>Case</u> T has ≥ 2 rows

$T \leftarrow i$ is obtained by

- insert i into row 1 as above
- whatever is bumped from row 1, insert into row 2 as above
- row 2 ... row 3 ...
- ... ...

By constr $T \leftarrow i$ is semi standard

Ex Fa

$$T = \begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & & \\ 3 & 3 & & & & & \\ 4 & & & & & & \end{array}$$

$$T \leftarrow 1 = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & 2 & 2 & 3 & 4 & & \\ 3 & 3 & 3 & & & & \\ 4 & & & & & & \end{array}$$

Lem For a semistandard tableau T and $1 \le i \le n$

Consider $T \leftarrow i$

Consider any entry of T, say $T(r,k)$

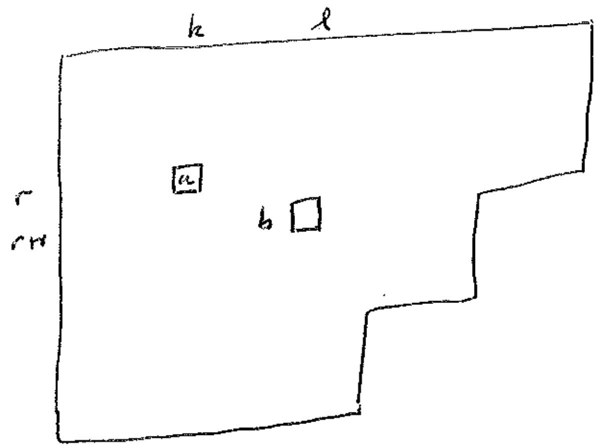
Assume $T(r,k)$ is bumped to row $r+1$ and col l.

Then $l \le k$.

pf Suppose $l > k$. Write

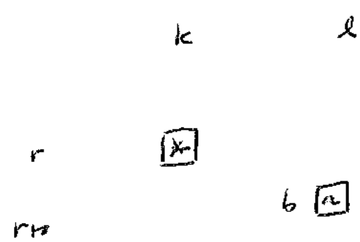
$a = T(r,k)$ $\qquad b = T(r+1, l-1)$

T :



$a < b$ since T is s. standard

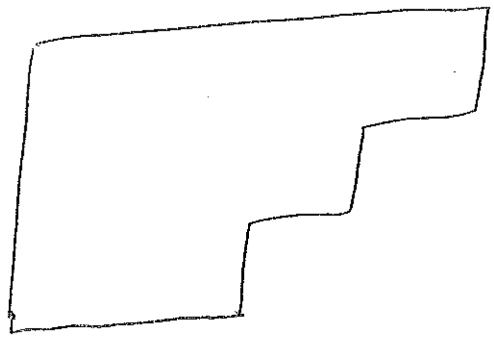
$T \leftarrow i$



$b \le a$ by bumping construction, cont. $\square$

Given semi standard tableau T shp λ

T :

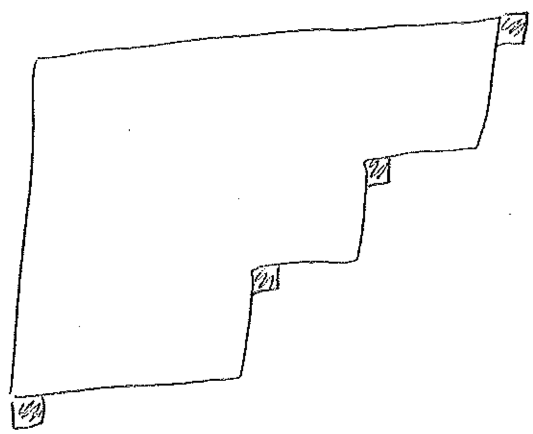


Given $1 \le i \le n$

$T \leftarrow i$ has shp μ

μ / λ is single box, called new location where is it?

$T \leftarrow i$:



▨ = poss new location

let Ω = set of locations for which

T, $T \leftarrow i$ have different entries

Ω is a vertical strip by prev lem

Ω called <u>bumping path</u>

Entries in $T \leftarrow i$ down bumping path strictly increase

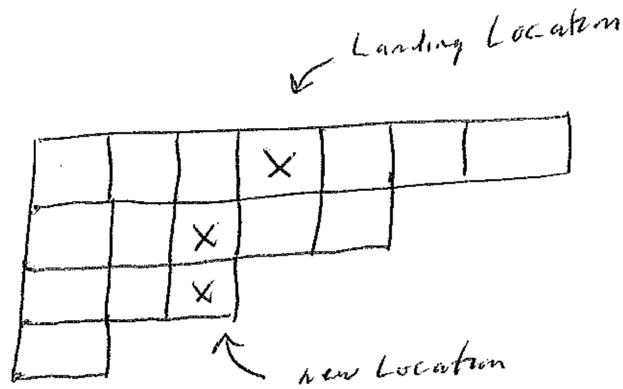
Ω contains unique box in row 1, called

<u>Landing Location</u>

Ex

$$T = \begin{array}{lllllll} 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 \\ 3 & 3 \\ 4 \end{array}$$

$$T \leftarrow 1 = \begin{array}{lllllll} 1 & 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 3 \\ 4 \end{array}$$



$\boxed{x} \in \Omega$