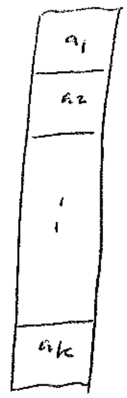
Lecture 31 Wednesday Nov 13 11/13/19

For $\Phi = C_r$ $\quad 1 \le k \le r$

Given a good element $x \in \mathbb{B}^{\otimes k}$:

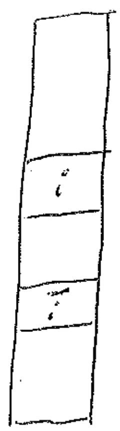
x :



$a_1 < a_2 < \cdots < a_k$

Assume $\exists\ i\ (1 \le i \le r)$ such that x contains

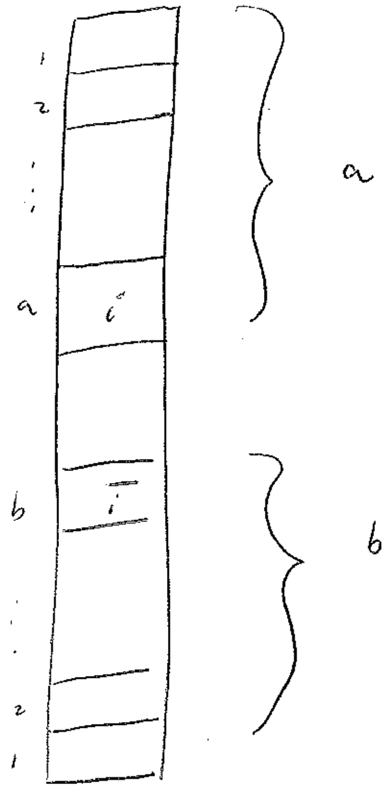
x :



Consider locations of $\boxed{i}$, $\boxed{\bar{i}}$

Write

X :



$\boxed{i}$ is a^{th} from top

$\boxed{\bar{i}}$ is b^{th} from bottom

Call X i-balanced whenever

$a + b \le i$

X is r-balanced since $a + b \le k \le r$.

Call X balanced whenever X is i-balanced for all i $(1 \le i \le r-1)$ st X contains $\boxed{i}$, $\boxed{\bar{i}}$

Thm For $\Phi = C_r$ and $1 \le k \le r$,

$$B\bar{\omega}_k = \left\{ x \in \mathbb{B}^{\otimes k} \;\middle|\; x \text{ is balanced and good} \right\}$$

$$\| \; C$$

pf Show

(i) C is disjoint union of connected components of $\mathbb{B}^{\otimes k}$

(ii) C contains no hw vector besides $\boxed{\begin{matrix}1\\2\\\vdots\\k\end{matrix}}$

pf (ii) Use earlier description of hw vectors

(i) For good $x, y \in \mathbb{B}^{\otimes k}$ and $i \in I$ st

$$x \xleftarrow{i} y$$

Show

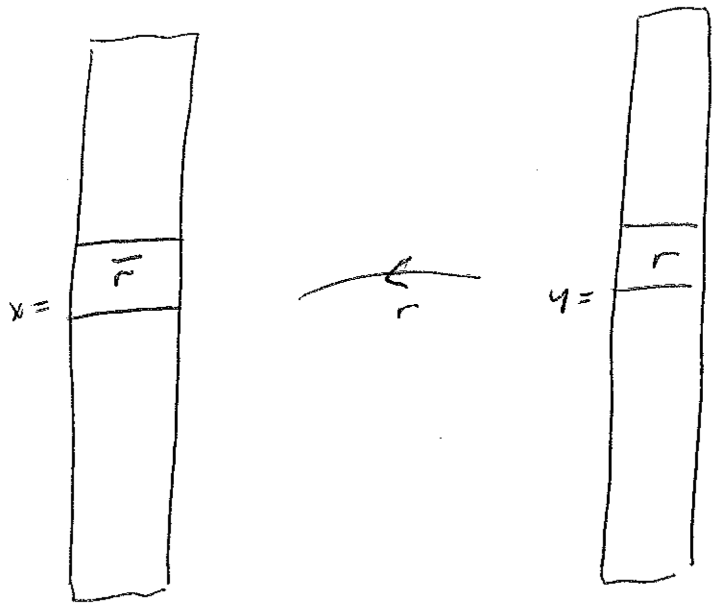
x balanced iff y balanced

Apply signature rule

Case $i = r$

$\bar{r}$ r
$<$ $>$

[showing all entries among $r, \bar{r}$.]



x is balanced iff y is balanced ✓

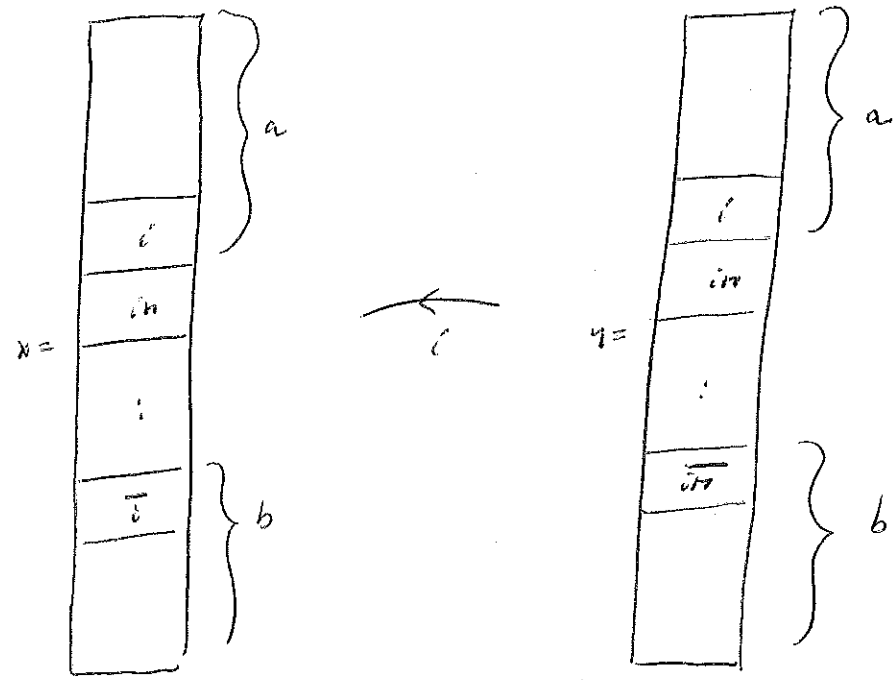
$\bar{i}$ $\overline{i+1}$ ... $i+1$ i
$<$ $>$ $<$ $>$

Case $1 \le i \le r-1$

The possibilities are I, II, III, IV below

[showing all entries among $i, i+1, \bar{i}, \overline{i+1}$]

I



Here

x is i-balanced

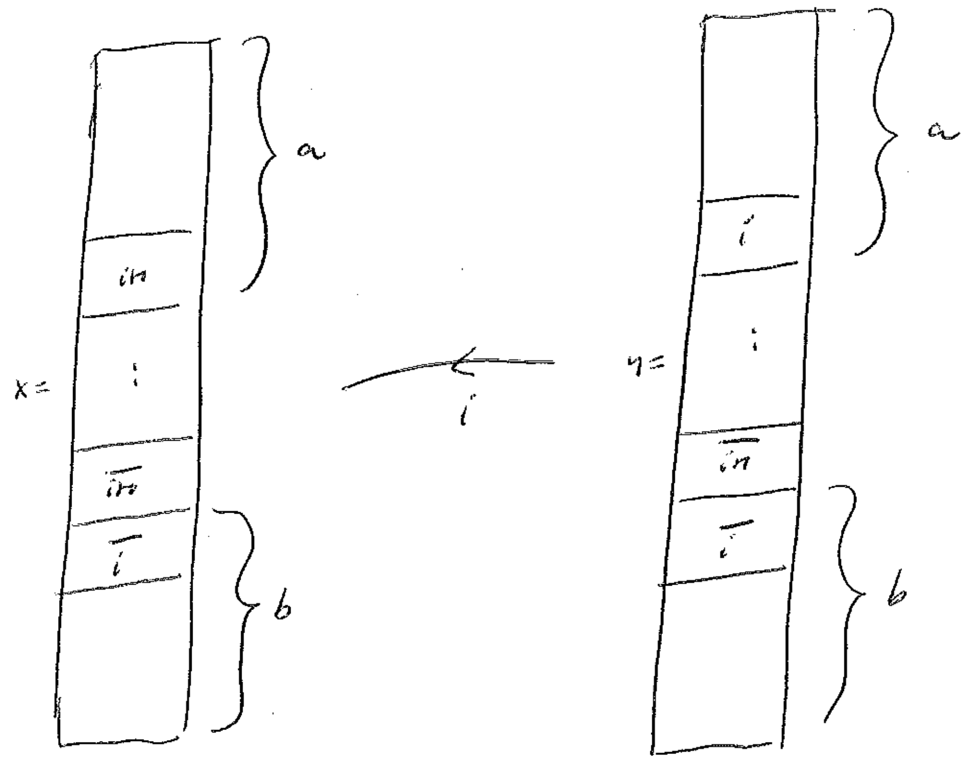
iff

$a+b \le i$

iff $\quad a+1+b \le i+1$

iff $\quad y$ is $(i+1)$-balanced

II



Here

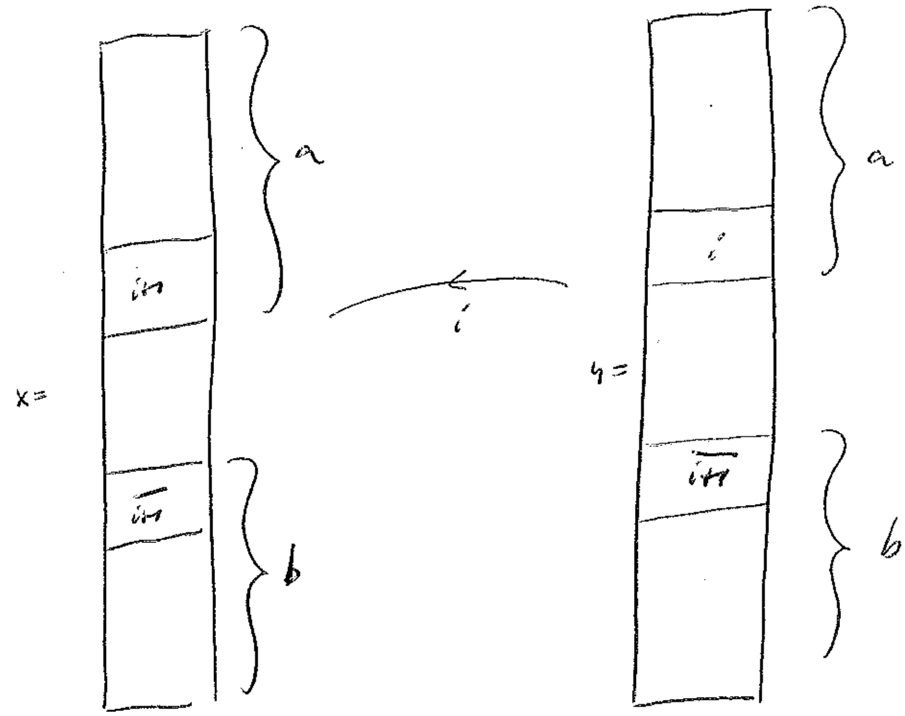
x is $(i+1)$-balanced

iff $a+b+1 \le i+1$

iff $a+b \le i$

iff y is i-balanced

III



Here show

x is (i+1)-balanced

show

$a + b \le i+1$

Consider

boxes above $\boxed{i+1}$ in x ✱

boxes below $\boxed{\overline{i+1}}$ in x ✱✱

$\boxed{i}$ not in $*$

$\boxed{\bar{i}}$ not in $**$

$\exists\ j$ st

$*$ contains $\boxed{j}$

and $**$ contains $\boxed{\bar{j}}$

else

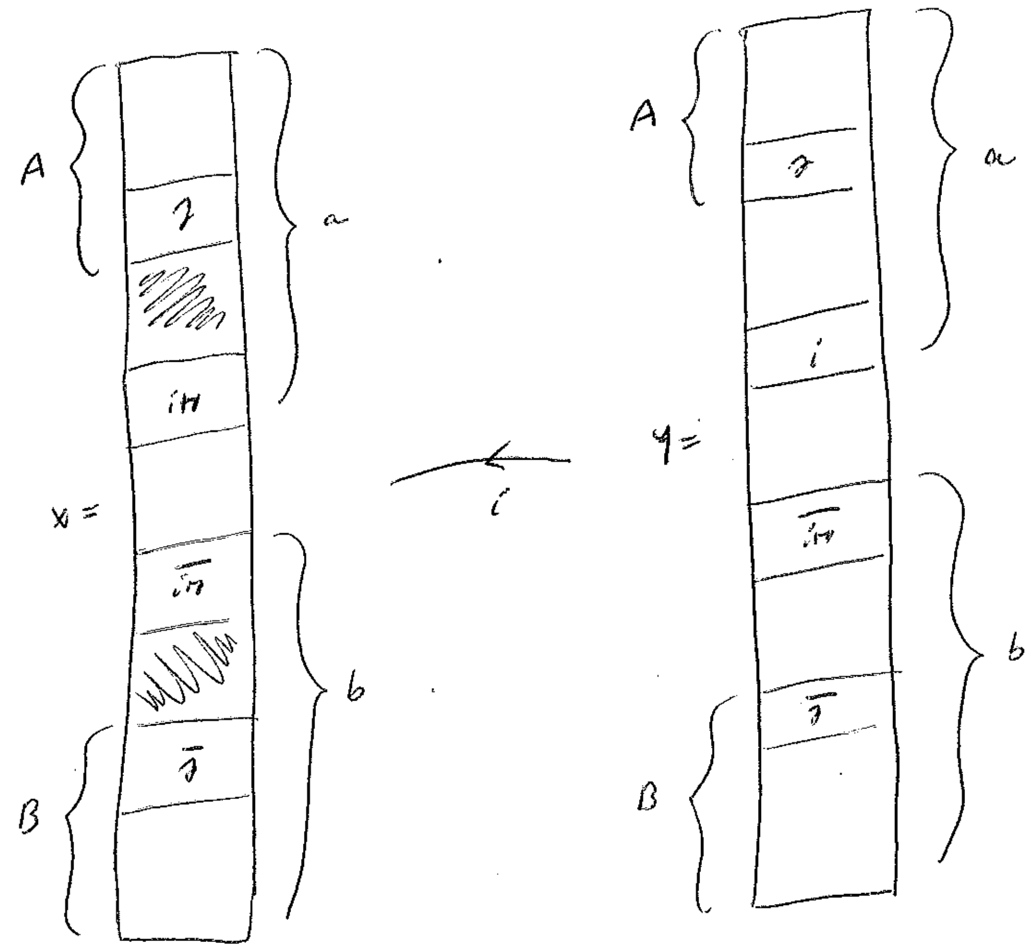
$a-1+b-1 \leq i-1$

forcing $a+b \leq i+1$

Take max'l such j

So $1 \leq j \leq i-1$

III, cont



WLOG

$A+B \le j$

else

neither of x, y is j-balanced

shaded entries $\in \{j+1, j+2, \dots, i-1\}$

shaded entries $\in \{\overline{j+1}, \overline{j+2}, \dots, \overline{i-1}\}$

$\nexists \; \ell$ s.t.

contains $\boxed{\ell}$

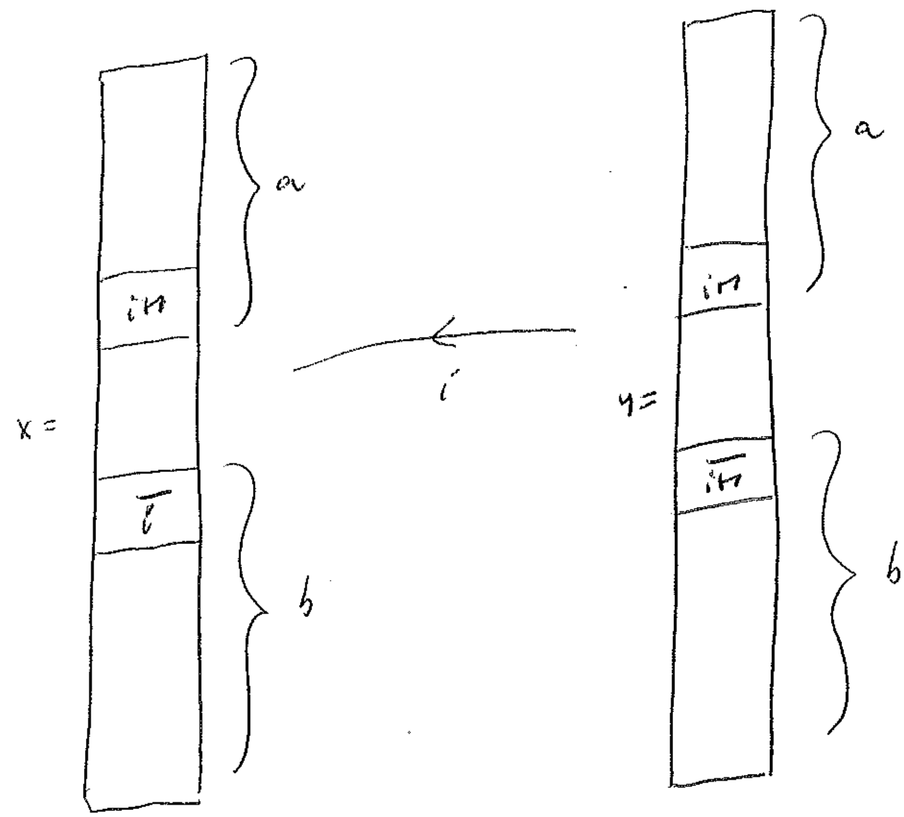
and contains $\boxed{\bar{\ell}}$

So
$$\underbrace{|\quad|}_{a-A-1} + \underbrace{|\quad|}_{b-B-1} \le \underbrace{|\{j+1, j+2, \dots, i-1\}|}_{i-1-(j+1)+1 \,=\, i-j-1}$$

$$a+b-A-B \le i-j+1$$

$$a+b \le i+A+B-j+1 \le i+1$$

IV



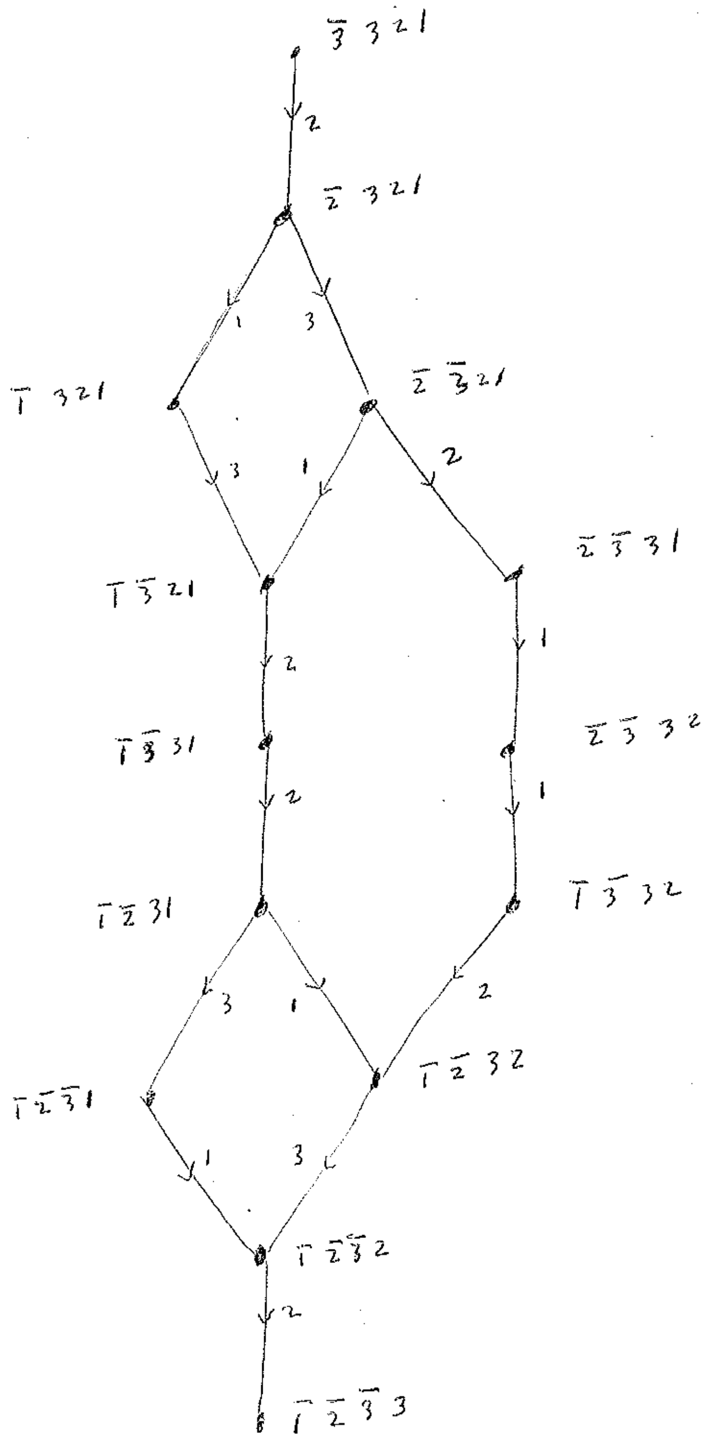
Here show

y is (im)-balanced.

Similar to III ex

$\square$

C_r $\qquad$ $r=3$


$\bar{3}\,3\,2\,1$
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$\bar{1}\,\bar{2}\,\bar{3}\,3$

We are done assuming $\Phi = C_r$

Consider $\Phi = B_r, D_r$

Write

$$\mathbb{B} = B_{\varpi_1} = \text{standard crystal for } \Phi$$

For $1 \le k \le r$ and $x \in \mathbb{B}^{\otimes k}$ write

$$x = \boxed{a_k} \otimes \cdots \otimes \boxed{a_1} \qquad a_i \in \Phi \text{ alphabet}$$

Call x <u>good</u> whenever

$$a_1 < a_2 < \cdots < a_k$$

except

- 0 can be repeated for B_r
- $r, \bar{r}$ can alternate for D_r

ex Some good elements for B_r

$$1\ 3\ 5\ 0\ 0\ 0\ \bar{4}\ \bar{2}$$

$$1\ 3\ 5\ 0\ \bar{4}\ \bar{2}$$

$$1\ 3\ 5\ \bar{4}\ \bar{2}$$

ex Some good elements for D_r

$$1\ 3\ 5\ r\ \bar{r}\ r\ \bar{r}\ \bar{4}\ \bar{2}$$

$$1\ 3\ 5\ r\ \bar{r}\ r\ \bar{4}\ \bar{2}$$

$$1\ 3\ 5\ \bar{r}\ r\ \bar{r}\ \bar{4}\ \bar{2}$$

$$1\ 3\ 5\ \bar{4}\ \bar{2}$$

For $\Phi = B_r, D_r$ and $1 \le k \le r$

Given a good element $x \in \mathcal{B}^{\otimes k}$

x is <u>balanced</u> whenever

x is i-balanced for all i $(1 \le i \le r-1)$ s.t. x contains $\boxed{i}$, $\boxed{\bar{i}}$

Thm For

$\Phi = B_r$ and $1 \le k \le r$

and for

$\Phi = D_r$ and $1 \le k \le r-1$,

$$\left\{ x \in \mathbb{B}^{\otimes k} \;\middle|\; x \text{ is good and balanced} \right\}$$

is equal to

$$\begin{cases} B\bar{\omega}_k & 1 \le k \le r-1 \\ B\,2\bar{\omega}_r & k = r \end{cases} \qquad \text{for } \Phi = B_r$$

and

$$\begin{cases} B\bar{\omega}_k & 1 \le k \le r-2 \\ B\,\bar{\omega}_{r-1}+\bar{\omega}_r & k = r-1 \end{cases} \qquad \text{for } \Phi = D_r$$

pf ex

$\square$

Next consider $\Phi = D_r$ and $k = r$

Thm For $\Phi = D_r$

$$B_{2\bar{\omega}_r} = \left\{ X \in \mathbb{B}^{\otimes r} \;\middle|\; \begin{array}{l} X \text{ is good and balanced and has form} \\ X = \begin{bmatrix} \left.\vdots\right\} a \\ r \\ \bar{r} \\ r \\ \bar{r} \\ \vdots \end{bmatrix} \qquad r-a \text{ even} \end{array} \right\}$$

$$B_{2\bar{\omega}_m} = \left\{ X \in \mathbb{B}^{\otimes r} \;\middle|\; \begin{array}{l} X \text{ is good and balanced and has form} \\ X = \begin{bmatrix} \left.\vdots\right\} a \\ r \\ \bar{r} \\ r \\ \bar{r} \\ \vdots \end{bmatrix} \qquad r-a \text{ odd} \end{array} \right\}$$

pf ex $\square$

Ex $\quad B_r \quad r=2$

signature rule

$$\boxed{\bar{1}} \xleftarrow{1} \boxed{\bar{2}} \xleftarrow{2} \boxed{0} \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

1 $\quad (\quad) \quad\quad\quad (\quad)$

2 $\quad \ll \quad >< \quad \gg$

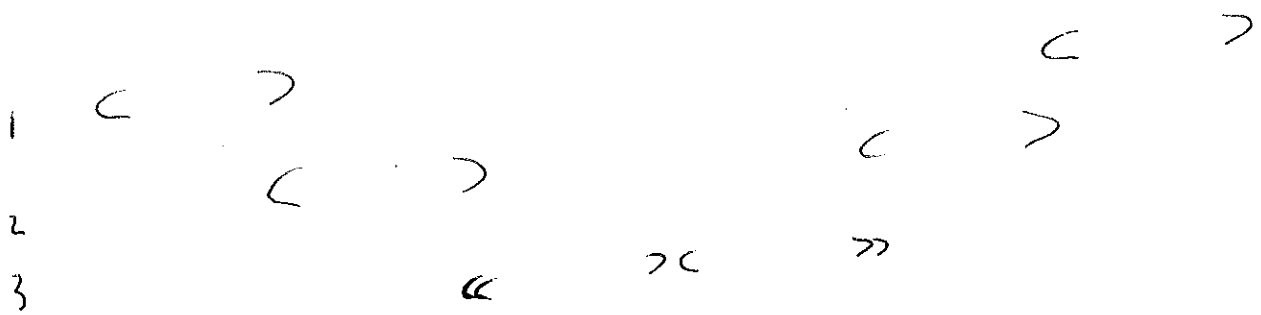
$\Phi = B_r$ $\qquad$ $r=2$ $\qquad$ $B_{2\varpi_r}$

$$
\begin{tikzcd}
 & \begin{array}{|c|}\hline 1 \\ 2 \\ \hline\end{array} \arrow[d, "2"] & \\
 & \begin{array}{|c|}\hline 1 \\ 0 \\ \hline\end{array} \arrow[dl, "2"'] \arrow[dr, "1"] & \\
\begin{array}{|c|}\hline 1 \\ \bar{2} \\ \hline\end{array} \arrow[d, "1"'] & & \begin{array}{|c|}\hline 2 \\ 0 \\ \hline\end{array} \arrow[d, "2"] \\
\begin{array}{|c|}\hline 2 \\ \bar{2} \\ \hline\end{array} \arrow[d, "1"'] & & \begin{array}{|c|}\hline 0 \\ 0 \\ \hline\end{array} \arrow[d, "2"] \\
\begin{array}{|c|}\hline 2 \\ \bar{1} \\ \hline\end{array} \arrow[dr, "2"'] & & \begin{array}{|c|}\hline 0 \\ \bar{2} \\ \hline\end{array} \arrow[dl, "1"] \\
 & \begin{array}{|c|}\hline 0 \\ \bar{1} \\ \hline\end{array} \arrow[d, "2"] & \\
 & \begin{array}{|c|}\hline \bar{2} \\ \bar{1} \\ \hline\end{array} &
\end{tikzcd}
$$

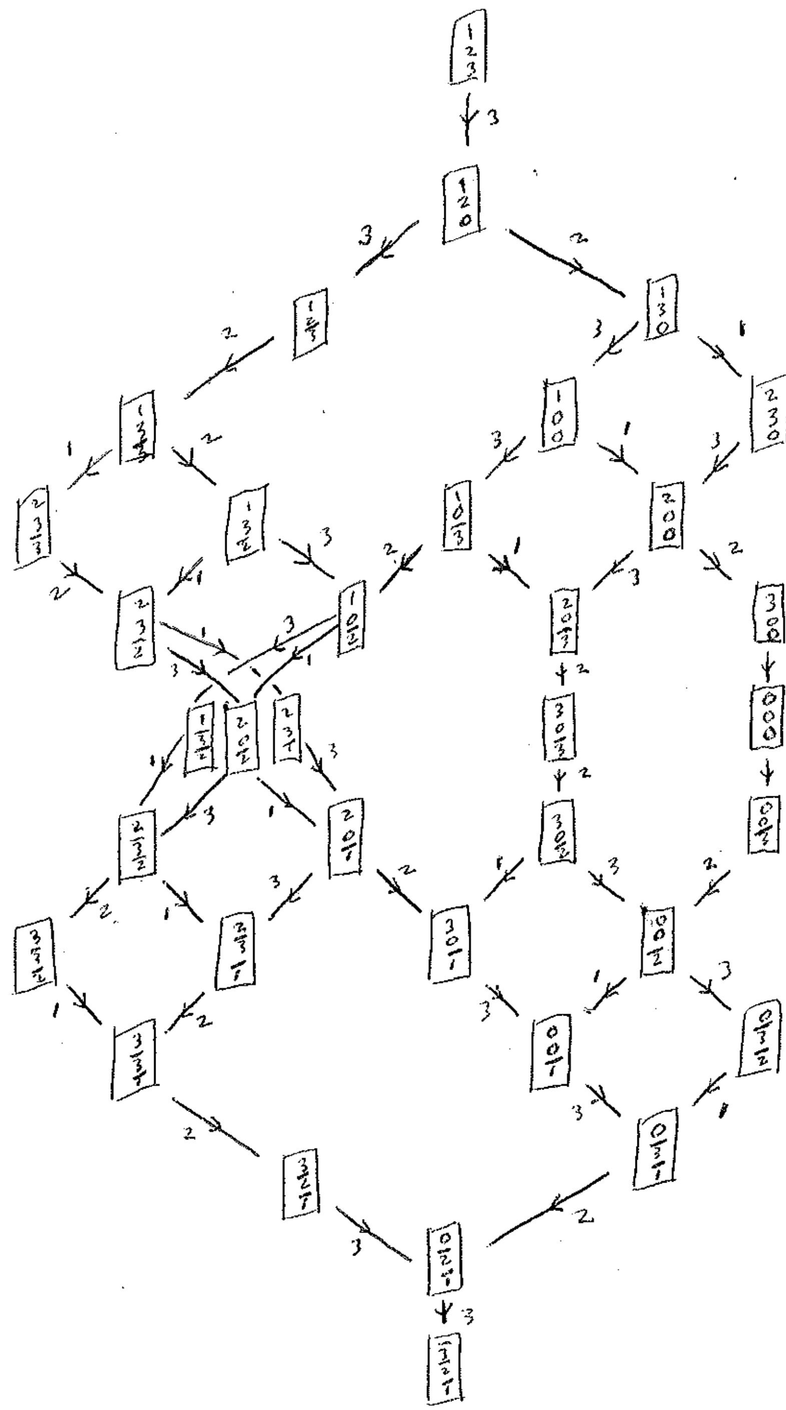
B_r $\qquad r=3$

Signature rule

$$\boxed{\bar 1} \xleftarrow{1} \boxed{\bar 2} \xleftarrow{2} \boxed{\bar 3} \xleftarrow{3} \boxed{0} \xleftarrow{3} \boxed{3} \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$



$\Phi = B_r$ $\quad B\,2\overline{\omega}_r$ $\quad r=3$ $\quad$ 11/13/14



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$\Phi = D_r$ $\quad r = 3$

Signature rule

$\mathbb{B}$:

$$\begin{tikzcd}[column sep=small]
 & & & \boxed{3} \arrow[dl, "3"'] & & \\
\boxed{\bar{1}} & \boxed{\bar{2}} \arrow[l, "1"] & & & \boxed{2} \arrow[ul, "2"'] \arrow[dl, "3"] & \boxed{1} \arrow[l, "1"] \\
 & & & \boxed{\bar{3}} \arrow[ull, "2"] & &
\end{tikzcd}$$

1: ( ) , , ( )

2: , (( )) ,

3: , ( ) ( ) ,

$\Phi = D_r$ $\quad r = 3 \quad k = 2 \quad B_{\varpi_2 + \varpi_3}$