Lecture 30 Monday Nov 11 11/11/19

Ch 6

For $\Phi = A_r$, $\Lambda = GL(r+1)$ $n = r+1$

For partition $\lambda \in \Lambda^+$

We identified B_λ with the set of semistandard tableaux shp λ with entries in alphabet $\{1, 2, \dots, n\}$

In particular, for $\lambda = \overline{\omega_k}$ a fundamental wt we have

$$B_\lambda : \quad \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \qquad 1 \le a_1 < a_2 < \cdots < a_k \le n$$

Next goal: Do something similar with "tableaux" for

B_r, C_r, D_r

Start with λ a fundamental wt $\overline{\omega_k}$

Exclude the minuscule wts

B_r $\quad \overline{\omega_r}$

D_r $\quad \overline{\omega_{r-1}}, \overline{\omega_r}$

Recall B_λ is iso the connected component of $\mathbb{B}^{\otimes k}$ ($\mathbb{B} = B_{\overline{\omega_1}}$ is standard crystal) that contains the hw element

$$u = \boxed{k} \otimes \boxed{k-1} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1}$$

Motivation: Consider $\Phi = C_2$

Recall

$$\mathbb{B} = B_{\varpi_1} : \quad \boxed{\bar 1} \xleftarrow{1} \boxed{\bar 2} \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

$B_{\varpi_2} \subseteq \mathbb{B}^{\otimes 2}$ contains hw element $\boxed{2} \otimes \boxed{1}$

We find

$$B_{\varpi_2} : \quad \boxed{\bar 1} \otimes \boxed{\bar 2} \xleftarrow{2} \boxed{\bar 1} \otimes \boxed{2} \xleftarrow{1} \boxed{\bar 2} \otimes \boxed{2} \xleftarrow{1} \boxed{\bar 2} \otimes \boxed{1} \xleftarrow{2} \boxed{2} \otimes \boxed{1}$$

Guided by A_r, use ColR to identify B_{ϖ_2} with

$$\begin{array}{|c|}\hline \bar 2 \\ \hline \bar 1 \\ \hline \end{array} \xleftarrow{2} \begin{array}{|c|}\hline 2 \\ \hline \bar 1 \\ \hline \end{array} \xleftarrow{1} \begin{array}{|c|}\hline 2 \\ \hline \bar 2 \\ \hline \end{array} \xleftarrow{1} \begin{array}{|c|}\hline 1 \\ \hline \bar 2 \\ \hline \end{array} \xleftarrow{2} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

So our alphabet should be

$$\{ 1, 2, \bar 1, \bar 2 \} \qquad *$$

In linear order

$$1 < 2 < \bar 2 < \bar 1$$

So

$$B_{\varpi_2} = \left\{ \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \end{array} \;\middle|\; a_1, a_2 \in *, \; a_1 < a_2, \; \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \end{array} \neq \begin{array}{|c|}\hline 1 \\ \hline \bar 1 \\ \hline \end{array} \right\}$$

So for B_r, C_r, D_r we expect

B_{ϖ_k} to consist of

- single column tableaux
- alphabet in bijection with the vertices for standard tableau.
- certain letter patterns forbidden
- hw vector is $\begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline k\\\hline\end{array}$

Alphabet in order

B_r: $\quad 1 < 2 < \cdots < r < 0 < \bar{r} < \cdots < \bar{2} < \bar{1}$

C_r: $\quad 1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{2} < \bar{1}$

D_r: $$1 < 2 < \cdots < r-1 \begin{array}{c} < \; r \; < \\ < \; \bar{r} \; < \end{array} \overline{r-1} < \cdots < \bar{2} < \bar{1}$$

(partial order only)

We next take a close look at C_r

Until further notice,

$\Phi = C_r$, $\mathbb{B} = B_{\varpi_1}$ standard crystal

$1 \le k \le r$

For $x \in \mathbb{B}^{\otimes k}$ write

$x = \boxed{a_k} \otimes \cdots \otimes \boxed{a_1}$ $a_i \in C_r$ alphabet

Call x <u>good</u> whenever

$a_1 < a_2 < \cdots < a_k$

LEM With above notation, set of good elements in $\mathbb{B}^{\otimes k}$ is a disjoint union of connected components of $\mathbb{B}^{\otimes k}$

pf For $x, y \in \mathbb{B}^{\otimes k}$ and $i \in I$ st

$x \xleftarrow{i} y$

show x good iff y good.

Apply signature rule

Case $1 \le i \le r-1$

$$\begin{array}{cccccccccccccc} \bar{1} & \bar{2} & \cdots & \bar{i} & \overline{i+1} & \cdots & \bar{r} & r & \cdots & i+1 & i & \cdots & 2 & 1 \\ & & & < & > & & & & & < & > & & & \end{array}$$

WLOG ignore all entries in x, y besides

$\bar{i}, \overline{i+1}, i+1, i$

List the i-chains in $\mathbb{B}^{\otimes k}$ that contain a good element

$\bar{i}\ \ \overline{i+1}\ \ i+1\ \ i$

$\bar{i}\ \ i+1\ \ i \xleftarrow[i]{} \overline{i+1}\ \ i+1\ \ i$

$\bar{i}\ \ \overline{i+1}\ \ i+1 \xleftarrow[i]{} \bar{i}\ \ \overline{i+1}\ \ i$

$\bar{i}\ \ i+1 \xleftarrow[i]{} \overline{i+1}\ \ i+1 \xleftarrow[i]{} \overline{i+1}\ \ i$

$i+1 \xleftarrow[i]{} i$

$\bar{i} \xleftarrow[i]{} \overline{i+1}$

$i+1\ \ i$

$\bar{i}\ \ i$

$\bar{i}\ \ \overline{i+1}$

ϕ

For each chain all the elements are good

So x good iff y good

Case $i=r$

$$\bar{1}\ \bar{2} \cdots \underset{<}{\bar{r}}\ \underset{>}{r} \cdots 2\,1$$

WLOG ignore all entries in x, y besides $\bar{r}, r$

List the r-chains in $\mathbb{B}^{\otimes k}$ that contain a good element

$$\overset{\bar{r}r}{\bullet}$$

$$\overset{\bar{r}}{\circ} \xleftarrow{\ r\ } \overset{r}{\circ}$$

$$\overset{\phi}{\bullet}$$

For each chain above all elements are good

So x good iff y good $\square$

So far,

$$\{\text{good elements of } \mathbb{B}^{\otimes k}\} = \text{disjoint union of connected components of } \mathbb{B}^{\otimes k} \qquad \ast$$

one of these is B_{wt}

To clarify the decomp $\ast$, next find good elements in $\mathbb{B}^{\otimes k}$ that are highest weights.

LEM For a good element

$$x = \boxed{a_k} \otimes \cdots \otimes \boxed{a_1}$$

in $\mathbb{B}^{\otimes k}$, x is hw iff $a_k, \ldots, a_1$ listed below

k	$a_k \cdots a_1$
2	$2\ 1$
	$\bar{1}\ 1$
3	$3\ 2\ 1$
	$\bar{2}\ 2\ 1$
4	$4\ 3\ 2\ 1$
	$\bar{3}\ 3\ 2\ 1$
	$\bar{1}\ \bar{2}\ 2\ 1$
5	$5\ 4\ 3\ 2\ 1$
	$\bar{4}\ 4\ 3\ 2\ 1$
	$\bar{2}\ \bar{3}\ 3\ 2\ 1$
6	$6\ 5\ 4\ 3\ 2\ 1$
	$\bar{5}\ 5\ 4\ 3\ 2\ 1$
	$\bar{3}\ \bar{4}\ 4\ 3\ 2\ 1$
	$\bar{1}\ \bar{2}\ \bar{3}\ 3\ 2\ 1$
$\vdots$	$\vdots$

pf use prev description of i-chains for $1 \le i \le r$ $\square$

ex $k = 3$ Find connected component of $\mathbb{B}^{\otimes 3}$

containing $\bar{2}\ 2\ 1$

signature rule

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\cdots$	$\bar{r}$	r	$\cdots$	3	2	1
1	$<$	$>$							$<$	$>$
2		$<$	$>$					$<$	$>$	
3			$<$	$>$			$<$	$>$		
$\vdots$				$\ddots$			$\cdots$			
r					$<$	$>$				

$$
\begin{array}{c}
\bar{2}\ 2\ 1 \\
\downarrow 1 \\
\bar{1}\ 2\ 1 \\
\downarrow 2 \\
\bar{1}\ 3\ 1 \\
\downarrow 3 \\
\bar{1}\ 4\ 1 \\
\downarrow \\
\vdots \\
\downarrow \\
\bar{1}\ r\ 1 \\
\downarrow r \\
\bar{1}\ \bar{r}\ 1 \\
\downarrow r-1 \\
\bar{1}\ \overline{r-1}\ 1 \\
\downarrow \\
\vdots \\
\downarrow 2 \\
\bar{1}\ \bar{2}\ 1 \\
\downarrow 1 \\
\bar{1}\ \bar{2}\ 2
\end{array}
$$

drop notation
$\square$, $\otimes$

Above elements excluded from B_{ϖ_3}

We need for $\Phi = C_r$ $(r \geq 3)$,

$$B_{\varpi_3} = \left\{ \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline a_3 \\ \hline \end{array} \;\middle|\; \begin{array}{l} a_1, a_2, a_3 \in C_r\text{-alphabet} \\ a_1 < a_2 < a_3 \end{array} \quad \begin{array}{c} \text{forbidden patterns} \\ \begin{array}{|c|}\hline 1 \\ \hline * \\ \hline \bar{1} \\ \hline \end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline * \\ \hline \end{array} \quad \begin{array}{|c|}\hline * \\ \hline 2 \\ \hline \bar{2} \\ \hline \end{array} \end{array} \right\}$$

$* = \text{arb}$

ex $k=4$ For $\Phi = C_r$ $r \geq 4$

natural to guess that

$$B_{\overline{\omega_4}} = \left\{ \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline a_3 \\ \hline a_4 \\ \hline \end{array} \;\middle|\; \begin{array}{l} a_1, a_2, a_3, a_4 \in C_r\text{-alphabet} \ldots \\ a_1 < a_2 < a_3 < a_4 \\ \\ \begin{array}{|c|}\hline 1 \\ \hline * \\ \hline * \\ \hline \bar{1} \\ \hline \end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline * \\ \hline \bar{2} \\ \hline * \\ \hline \end{array} \quad \begin{array}{|c|}\hline * \\ \hline 2 \\ \hline * \\ \hline \bar{2} \\ \hline \end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline \bar{3} \\ \hline * \\ \hline * \\ \hline \end{array} \quad \begin{array}{|c|}\hline * \\ \hline 3 \\ \hline \bar{3} \\ \hline * \\ \hline \end{array} \quad \begin{array}{|c|}\hline * \\ \hline * \\ \hline 3 \\ \hline \bar{3} \\ \hline \end{array} \end{array} \quad \text{Forbidden patterns} \right\}$$

Next we clarify the forbidden patterns