

Ch 6

Lecture 30 Monday Nov 11

11/11/19

For $\mathbb{F} = \mathbb{A}_r$, $\Lambda = GL(r+1)$ $n = r+1$

For partition $\lambda \in \Lambda^+$

We identified B_λ with the set of semi-standard tableaux shp λ with entries in alphabet $\{1, 2, \dots, n\}$

In particular, for $\lambda = \bar{w}_k$ a fundamental wt we have

$B_\lambda:$

a_1
a_2
\vdots
a_k

$1 \leq a_1 < a_2 < \dots < a_k \leq n$

Next goal: Do something similar with "tableaux" for

B_r, C_r, D_r

Start with λ a fundamental wt \bar{w}_k

Exclude the minuscule wts

B_r \bar{w}_r

D_r \bar{w}_{r-1}, \bar{w}_r

Recall B_λ is iso the connected component of

$B^{\otimes k}$ ($B = B_{w_1}$ is standard crystal) that

contains the h.c. element

$$u = \boxed{k} \otimes \boxed{1, \dots, k-1} \otimes \dots \otimes \boxed{2} \otimes \boxed{1}$$

Motivation: Consider $\mathbb{F} = \mathbb{C}_2$

Recall

$$\mathbb{B} = B\bar{w}_1 : \begin{array}{c} \boxed{1} \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{2} \\ \leftarrow_2 \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow_1 \end{array}$$

$B\bar{w}_2 \subseteq \mathbb{B}^{\otimes 2}$ contains hw element $\begin{array}{c} \boxed{2} \\ \otimes \end{array} \begin{array}{c} \boxed{1} \end{array}$

We find

$$B\bar{w}_2 : \begin{array}{c} \boxed{1} \\ \otimes \end{array} \begin{array}{c} \boxed{2} \\ \leftarrow_2 \end{array} \begin{array}{c} \boxed{1} \\ \otimes \end{array} \begin{array}{c} \boxed{2} \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{2} \\ \otimes \end{array} \begin{array}{c} \boxed{2} \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{2} \\ \otimes \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow_2 \end{array} \begin{array}{c} \boxed{2} \\ \otimes \end{array} \begin{array}{c} \boxed{1} \end{array}$$

Guided by A_1 , use ColR to identify $B\bar{w}_2$ with

$$\begin{array}{c} \boxed{2} \\ \boxed{1} \end{array} \begin{array}{c} \leftarrow_2 \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{2} \\ \boxed{1} \end{array} \begin{array}{c} \leftarrow_1 \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{2} \\ \boxed{2} \end{array} \begin{array}{c} \leftarrow_1 \\ \leftarrow_1 \end{array} \begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} \begin{array}{c} \leftarrow_2 \\ \leftarrow_2 \end{array} \begin{array}{c} \boxed{1} \\ \boxed{2} \end{array}$$

So our alphabet should be

$$\{ 1, 2, \bar{1}, \bar{2} \} \quad *$$

In linear order

$$1 < 2 < \bar{2} < \bar{1}$$

So

$$B\bar{w}_2 = \left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \mid a_1, a_2 \in *, a_1 < a_2, \begin{array}{c} a_1 \\ a_2 \end{array} \neq \begin{array}{c} 1 \\ \bar{1} \end{array} \right\}$$

So for B_n, C_n, D_n we expect

B_{2n} to consist of

- single column tableaux
- alphabet in bijection with the vertices of standard tableaux.
- certain letter patterns forbidden
- hw vector is

1
2
⋮
k

Alphabet in order

$$B_r: 1 < 2 < \dots < r < \bar{0} < \bar{1} < \dots < \bar{z} < \bar{T}$$

$$C_r: 1 < 2 < \dots < r < \bar{r} < \dots < \bar{z} < \bar{T}$$

$$D_r: 1 < 2 < \dots < r \rightarrow \begin{array}{c} \swarrow \quad \searrow \\ r \quad \bar{r} \\ \swarrow \quad \searrow \\ \bar{r} \quad r \end{array} \rightarrow \bar{r} < \dots < \bar{z} < \bar{T}$$

(partial order only)

We next take a close look at C_r

Until otherwise,

11/11/19
5

$$\Phi = C_r, \quad B = B\bar{w}, \quad \text{standard crystal}$$

$1 \leq k \leq r$

For $x \in B^{\otimes k}$ write

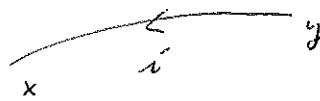
$$x = \boxed{a_{1k}} \otimes \dots \otimes \boxed{a_{rk}} \quad a_r \in C_r \text{ alphabet}$$

Call x good whenever

$$a_1 < a_2 < \dots < a_k$$

LEM With above notation, set of good elements in $B^{\otimes k}$ is a disjoint union of connected components of $B^{\otimes k}$

pf For $x, y \in B^{\otimes k}$ and $i \in I$ st



show x good iff y good.

Apply signature rule

Case $1 \leq i \leq r$

$$\bar{1} \ \bar{2} \ \dots \ \bar{i} \ \bar{i+1} \ \dots \ \bar{r} \ \dots \ i+1 \ \dots \ 2r$$

$< \quad >$ $< \quad >$

WLOG ignore all entries in x, y besides

$$\bar{i}, \bar{i+1}, i+1, i$$

List the i -chains in B^{ok} that contain a good element

$$\bar{i} \quad \bar{in} \quad in \quad i$$

$$\bar{i} \quad in \quad i \quad \xleftarrow{i} \quad \bar{in} \quad in \quad i$$

$$\bar{i} \quad \bar{in} \quad in \quad \xleftarrow{i} \quad \bar{i} \quad \bar{in} \quad i$$

$$\bar{i} \quad in \quad \xleftarrow{i} \quad \bar{in} \quad in \quad \xleftarrow{i} \quad \bar{in} \quad i$$

$$in \quad \xleftarrow{i} \quad i$$

$$\bar{i} \quad \xleftarrow{i} \quad \bar{in}$$

$$in \quad i$$

$$\bar{i} \quad i$$

$$\bar{i} \quad \bar{in}$$

$$\phi$$

For each chain all the elements are good

So x good iff y good

Case $i=r$

$$\bar{r} \bar{z} \dots \bar{r} r \dots z1$$

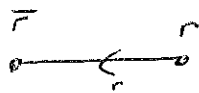
$$\langle \rangle$$

WLOG ignore all entries in x, y besides \bar{r}, r

List the r -chains in \mathbb{B}^{nk} that contain a good element

$\bar{r} r$

\circ



ϕ

\circ

For each chain above all elements are good
So x good iff y good

□

So far,

$$\left\{ \text{good elements of } B^{\otimes k} \right\} = \text{disjoint union of connected components of } B^{\otimes k}$$

*

↑
one of these is $B^{\otimes k}$

To clarify the decomp *, next find good elements in $B^{\otimes k}$ that are highest weights.

LEM For a good element

$$x = \boxed{a_k} \otimes \dots \otimes \boxed{a_1}$$

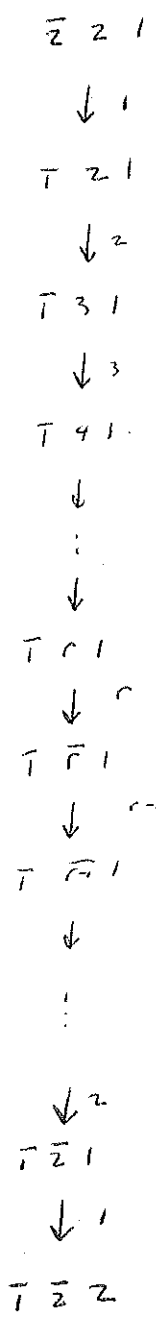
in $B^{\otimes k}$, x is h.w. iff a_k, \dots, a_1 listed below

k	$a_k \dots a_1$
2	2 1 1 1
3	3 2 1 2 2 1
4	4 3 2 1 3 3 2 1 1 2 2 1
5	5 4 3 2 1 4 4 3 2 1 2 3 3 2 1
6	6 5 4 3 2 1 5 5 4 3 2 1 3 4 4 3 2 1 1 2 3 3 2 1
⋮	⋮

pf use prev description of i-chains of 1515r



ex $k=3$ Find connected component of $B^{\otimes 3}$ containing $\bar{2} \bar{2} 1$



signature rule

	$\bar{1}$	$\bar{2}$	$\bar{3}$	\bar{r}	\dots	3	2	1
1	$\langle \rangle$							$\langle \rangle$
2		$\langle \rangle$						$\langle \rangle$
3			$\langle \rangle$					$\langle \rangle$
\vdots				\dots	\dots			
r				$\langle \rangle$				

drop notation
□, ⊗

Above elements excluded from $B^{\otimes 3}$

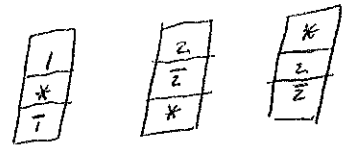
11 | 11 | 19
"

Wence for $\Phi = C_r$ ($r \geq 3$),

$$B_{\overline{w}_3} = \left\{ \begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline a_3 \\ \hline \end{array} \right\} /$$

$a_1, a_2, a_3 \in C_r$ -alphabet
 $a_1 < a_2 < a_3$

forbidden patterns



* = arb

11/11/19
12

ex $k=4$ $F_n \bar{\Phi} = C_r$ $r \geq 4$

natural to guess that

$B_{\bar{w}_4} = \left\{ \begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline a_3 \\ \hline a_4 \\ \hline \end{array} \right\} /$

$a_1, a_2, a_3, a_4 \in C_r$ - alphabet ---
 $a_1 < a_2 < a_3 < a_4$

Forbidden patterns

1	2	x	?	x	x
x	y	z	z	x	x
x	z	x	x	z	z
i	x	z	x	x	z

Next we clarify the forbidden patterns