Lecture 3 Monday Sept 9

For $w \in W$

the <u>length</u> of w is the minimal $\ell \in \mathbb{N}$ st

w = product of ℓ simple reflections

Write
$\ell(w)$ = length of w

LEM 7 For $w \in W$

$$\ell(w) = |w(\Phi^+) \cap \Phi^-|$$

pf Define

$$L(w) = |w(\Phi^+) \cap \Phi^-|$$

Show $\ell(w) = L(w)$.

Write w as a product of $\ell(w)$ simple reflections.
Each factor moves one element of Φ^+ to Φ^-.
Some of these elements might return to Φ^+, so

$\ell(w) \geq L(w)$.

Call w <u>good</u> whenever $\ell(w) = L(w)$

1 is good

s_i good $\quad i \in I$

Assume $\exists$ element of W that is not good.

WLOG $\quad w$ is not good with minimal length.

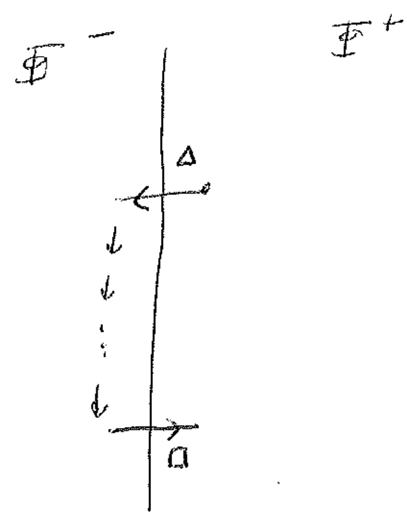
Write

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \qquad \ell = \ell(w) \geq 2$$

Must have

$$w = s_{i_1} \cdots \square \cdots \triangle \cdots s_{i_\ell}$$

with



Since ℓ is minimal,

$$w = \boxed{s_{i_1}} \; s_{i_2} \cdots s_{i_{\ell-1}} \; s_{i_\ell}$$

Define

$$\theta = s_{i_2} \cdots s_{i_{\ell-1}}$$

show

$$\begin{array}{c} w \overset{?}{=} \theta \\ \| \\ s_{i_1} \theta s_{i_\ell} \end{array}$$

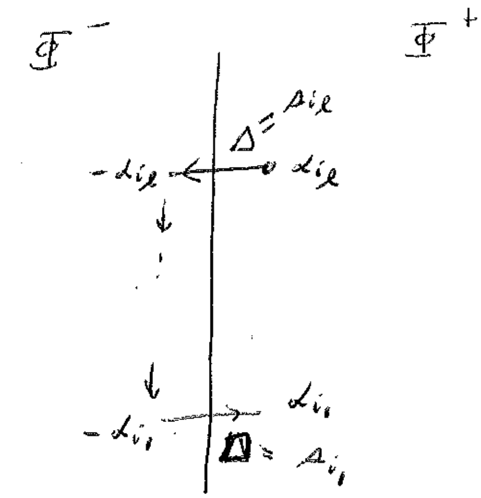
show

$$\underbrace{\theta \underbrace{s_{i_\ell}}_{= r_{\alpha_{i_\ell}}} \theta^{-1}}_{= r_{\theta(\alpha_{i_\ell})}} \overset{?}{=} \underset{= r_{\alpha_{i_1}}}{s_{i_1}} \qquad \theta \in O(V)$$

show

$$\theta(\alpha_{i_\ell}) = \alpha_{s_1}$$

We have



So $$w(\alpha_{i_\ell}) = \alpha_{i_1}$$

So $$s_{i_1}\,\theta\, s_{i_\ell}(\alpha_{i_\ell}) = \alpha_{i_1}$$

So $$\theta\, \underbrace{s_{i_\ell}(\alpha_{i_\ell})}_{= -\alpha_{i_\ell}} = \underbrace{s_{i_1}(\alpha_{i_1})}_{= -\alpha_{i_1}}$$

So $$\theta(\alpha_{i_\ell}) = \alpha_{i_1}$$ ✓

We have shown $w = \theta$

Now $w = \theta$ has length $\leq \ell - 2$ cont.

So all the elements of W are good. □

Fr $w \in W$ and $i \in I$.

compare

$$\ell(ws_i), \quad \ell(w)$$

Write

$$N = N(w) = \{ \alpha \in \Phi^+ / w(\alpha) \in \Phi^- \}$$

so $\ell(w) = |N|$

One checks:

Fr $\alpha_i \notin N$,

$$N(ws_i) = s_i(N) \cup \{\alpha_i\}$$

$$\ell(ws_i) = \ell(w) + 1$$

Fr $\alpha_i \in N$,

$$N(ws_i) = s_i\left(N - \{\alpha_i\}\right)$$

$$\ell(ws_i) = \ell(w) - 1$$

LEM 8 (i) $\exists$ unique $w_0 \in W$ st

$$w_0(\Phi^+) = \Phi^-$$

*

(ii) $\ell(w_0) = |\Phi^+|$

(iii) $w_0^2 = 1$.

Call w_0 the <u>longest element</u> of W

pf (i) Pick $w_0 \in W$ of maximal length

For $i \in I$, $w_0(\alpha_i) \in \Phi^-$

else $\ell(w_0 s_i) = \ell(w_0) + 1$

For $\alpha \in \Phi^+$ write

$$\alpha = \sum_{i \in I} a_i \alpha_i \qquad a_i \in \mathbb{N}$$

$$w_0(\alpha) = \sum_{i \in I} \underset{\geq 0}{a_i} \underset{\Phi^-}{w_0(\alpha_i)}$$

$$\in \Phi^-$$

So $w_0(\Phi^+) \subseteq \Phi^-$

$*$ follows since $|\Phi^+| = |\Phi^-|$

We showed w_0 exists.

Suppose w_0 not unique. Then

$$w_0(\Phi^\pm) = \Phi^\mp$$

$$w_0'(\Phi^\pm) = \Phi^\mp$$

So $w_0^{-1} w_0'(\Phi^+) = \Phi^+$

So $\ell(w_0^{-1} w_0') = 0$

so $w_0^{-1} w_0' = 1$

so $w_0 = w_0'$

(ii) By (i)

(iii) Since w_0^{-1} satisfies (ii). $\square$

Any product of simple reflections

$$s_{i_1} s_{i_2} \cdots s_{i_\ell} \quad *$$

has length $\leq \ell$. Call $*$ <u>reduced</u> whenever its length is ℓ.

Aside on Linear algebra

Given basis $\{v_i\}_{i=1}^n$ for Eucl space V.

Dual basis $\{w_i\}_{i=1}^n$ for V satisfies

$$\langle v_i, w_j \rangle = \delta_{ij} \qquad 1 \le i, j \le n.$$

We have

$$v_j = \sum_{i=1}^n \langle v_i, v_j \rangle w_i \qquad 1 \le j \le n$$

$$w_j = \sum_{i=1}^n \langle w_i, w_j \rangle v_i$$

LEM 9 With above notation. assume

$$\langle v_i, v_j \rangle \le 0 \text{ if } i \ne j \qquad (1 \le i, j \le n)$$

then

$$\langle w_i, w_j \rangle \ge 0 \qquad (1 \le i, j \le n)$$

pf Ind on n

$n = 1$ $\langle w_1, w_1 \rangle > 0$ ✓

$n > 1$: Write

$$H = \mathrm{Span}(v_1 \ldots v_{n-1})$$

Obs

$$V = H + \mathbb{R} w_n \qquad \text{(orth dir sum)}$$

Let $v_1^*, \ldots, v_{n-1}^*$ denote basis for H dual to $v_1 \ldots v_{n-1}$ rel $\langle \, , \rangle$

So $\langle v_i^*, w_n \rangle = 0$ $\quad 1 \le i \le n-1$

By ind $\langle v_i^*, v_j^* \rangle \ge 0$ $\quad 1 \le i,j \le n-1$

Fa $1 \le i \le n-1$

Compare v_i^*, w_i

$$\langle v_i^*, v_j \rangle = \delta_{ij} \qquad 1 \le j \le n-1$$
$$\langle w_i, v_j \rangle = \delta_{ij} \qquad \text{''}$$

$$\langle v_i^* - w_i, v_j \rangle = 0 \qquad \text{''}$$

$$\langle v_i^* - w_i, H \rangle = 0$$

$$v_i^* - w_i \in \mathbb{R} w_n$$

Write $v_i^* - w_i = \lambda_i w_n$

$$\langle v_i^*, v_n \rangle - \underbrace{\langle w_i, v_n \rangle}_{= 0} = \lambda_i \underbrace{\langle w_n, v_n \rangle}_{= 1}$$

$$\lambda_i = \langle v_i^*, v_n \rangle$$

So
$$w_i = v_i^* - \langle v_i^*, v_n \rangle w_n$$

Claim $\langle v_i^*, v_n \rangle \le 0$ $\quad 1 \le i \le n-1$

pf of
$$v_i^* = \sum_{j=1}^{n-1} \langle v_i^*, v_j^* \rangle v_j$$

so
$$\langle v_i^*, v_n \rangle = \sum_{j=1}^{n-1} \underbrace{\langle v_i^*, v_j^* \rangle}_{\substack{\text{|V}\\ 0}} \underbrace{\langle v_j, v_n \rangle}_{\substack{\text{|}\wedge\\ 0}}$$
$$\leq 0$$

For $1 \leq i, j \leq n-1$
$$\langle w_i, w_j \rangle = \left\langle v_i^* - \langle v_i^*, v_n \rangle w_n,\ v_j^* - \langle v_j^*, v_n \rangle w_n \right\rangle$$
$$= \underbrace{\langle v_i^*, v_j^* \rangle}_{\substack{\text{|V}\\ 0}} + \underbrace{\langle v_i^*, v_n \rangle}_{\substack{\text{|}\wedge\\ 0}} \underbrace{\langle v_j^*, v_n \rangle}_{\substack{\text{|}\wedge\\ 0}} \underbrace{\langle w_n, w_n \rangle}_{\substack{\vee\\ 0}}$$

$\checkmark$

$$\geq 0$$

For $1 \leq i \leq n-1$
$$\langle w_i, w_n \rangle = \left\langle v_i^* - \langle v_i^*, v_n \rangle w_n,\ w_n \right\rangle$$
$$= \underbrace{\langle v_i^*, w_n \rangle}_{\substack{\shortparallel\\ 0}} - \underbrace{\langle v_i^*, v_n \rangle}_{\substack{\text{|}\wedge\\ 0}} \underbrace{\langle w_n, w_n \rangle}_{\substack{\vee\\ 0}}$$
$$\geq 0$$

Also $\langle w_n, w_n \rangle > 0$

Result follows. $\square$

For the weight lattice Λ,

define a partial order $\geq$ on Λ st

for $\lambda, \mu \in \Lambda$

$\lambda \geq \mu$ whenever

$$\lambda - \mu = \sum_{i \in I} a_i \alpha_i \qquad a_i \geq 0$$

Also define

$$\Lambda^+ = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I \}$$

"dominant weights"

A wt $\lambda \in \Lambda$ is called <u>strictly dominant</u>

whenever $\langle \lambda, \alpha_i^\vee \rangle > 0 \quad \forall i \in I$

For $i \in I$ the ith <u>fundamental weight</u> ϖ_i

satisfies

$$\langle \varpi_i , \alpha_j^\vee \rangle = \delta_{ij} \qquad j \in I$$

ϖ_i is unique if Λ is s.s. Else we pick

a convenient ϖ_i

Assume Λ is s.s.

So Φ spans V

V has basis $\{\alpha_i\}_{i\in I}$ and a basis $\{\alpha_i^\vee\}_{i\in I}$

the $\{\varpi_i\}_{i\in I}$ form a basis for V dual to $\{\alpha_i^\vee\}_{i\in I}$ rel $\langle , \rangle$

LEM 10 Assume Λ is s.s.

then $\lambda \geq 0 \quad \forall \lambda \in \Lambda^+$

pf By linear alg

$$\lambda = \sum_{j \in I} \underbrace{\langle \lambda, \alpha_j^\vee \rangle}_{\geq 0} \bar{\omega}_j$$

For $j \in I$ show $\bar{\omega}_j \geq 0$

$$\bar{\omega}_j = \sum_{i \in I} \langle \bar{\omega}_i, \bar{\omega}_j \rangle \alpha_i^\vee \qquad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \qquad *$$

Recall

$$\langle \alpha_i, \alpha_j \rangle \leq 0 \quad \text{if } i \neq j \qquad i, j \in I$$

So $\langle \alpha_i^\vee, \alpha_j^\vee \rangle \leq 0$ ...

So $\langle \bar{\omega}_i, \bar{\omega}_j \rangle \geq 0 \qquad i, j \in I$

Now by $*$

$\bar{\omega}_j$ is non neg linear comb of $\{\alpha_i\}_{i \in I}$

$\square$