Lecture 29 Fri Nov 8

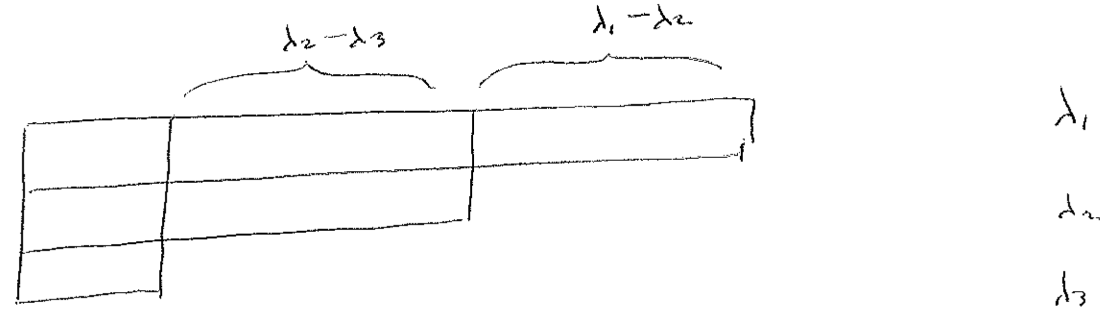
The crystal B_∞

(For $\Phi = A_2$, $\Lambda = GL(3)$ only)

<u>Motivation</u> Consider

B_λ partition $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda^+$

λ :



Roughly speaking,

B_∞ is obtained from B_λ by letting

$$\lambda_1 - \lambda_2 \longrightarrow \infty,$$
$$\lambda_2 - \lambda_3 \longrightarrow \infty$$

To be more precise, consider

B_λ $\lambda = (7, 3, 0)$ [See Lec 16]

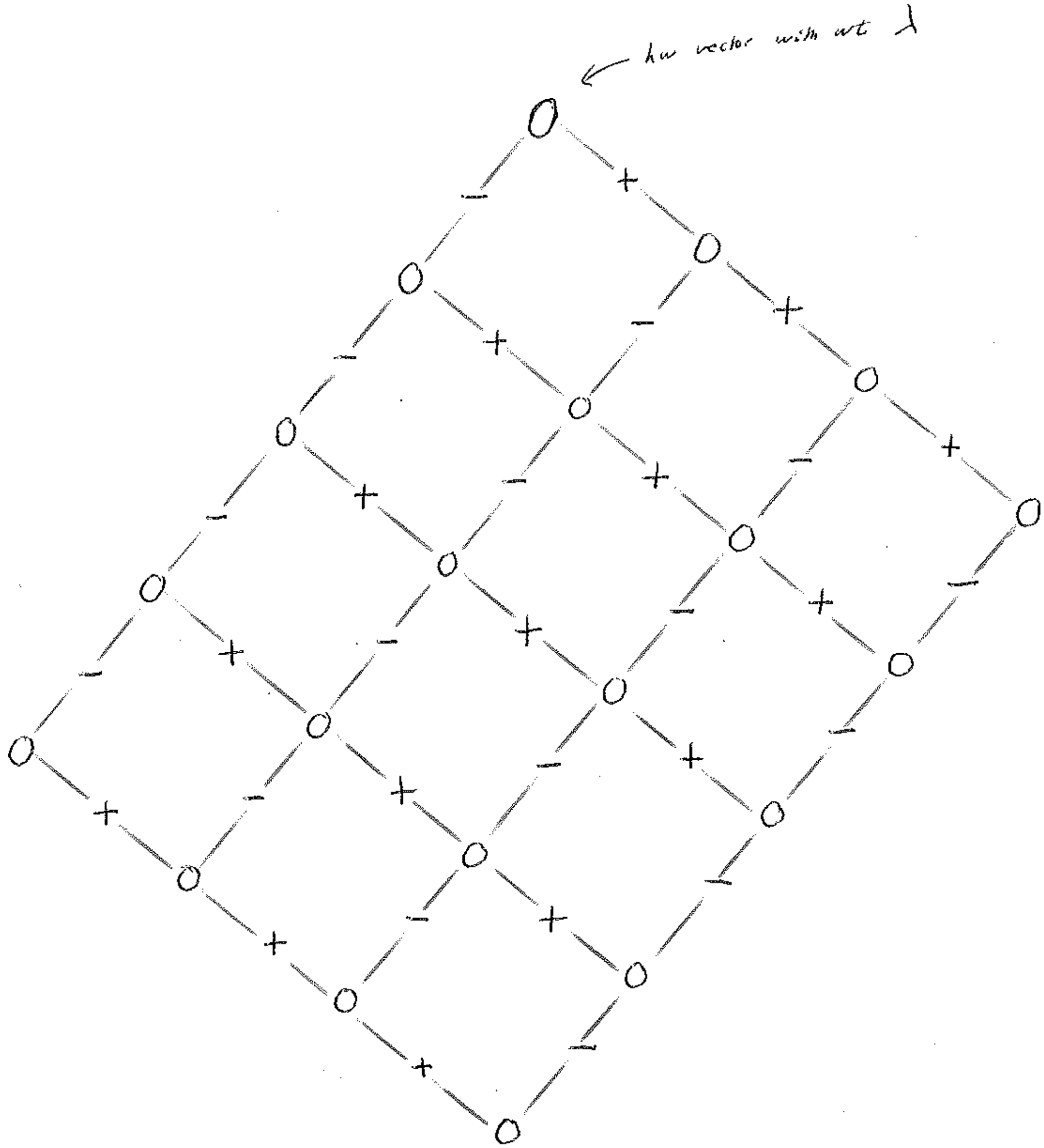
Recall level function

$$\text{level}(x) = \frac{\langle \text{wt}(x) - \lambda, \alpha_1^\vee - \alpha_2^\vee \rangle}{3} \qquad x \in B_\lambda$$

$\Phi = A_2$, $\Lambda = GL(3)$, B_λ, $\lambda = (7,3,0)$

Showing "subcrystal" induced on levels $-1, 0, 1$

We see a "rectangle" with dimensions $4, 3$



Key:

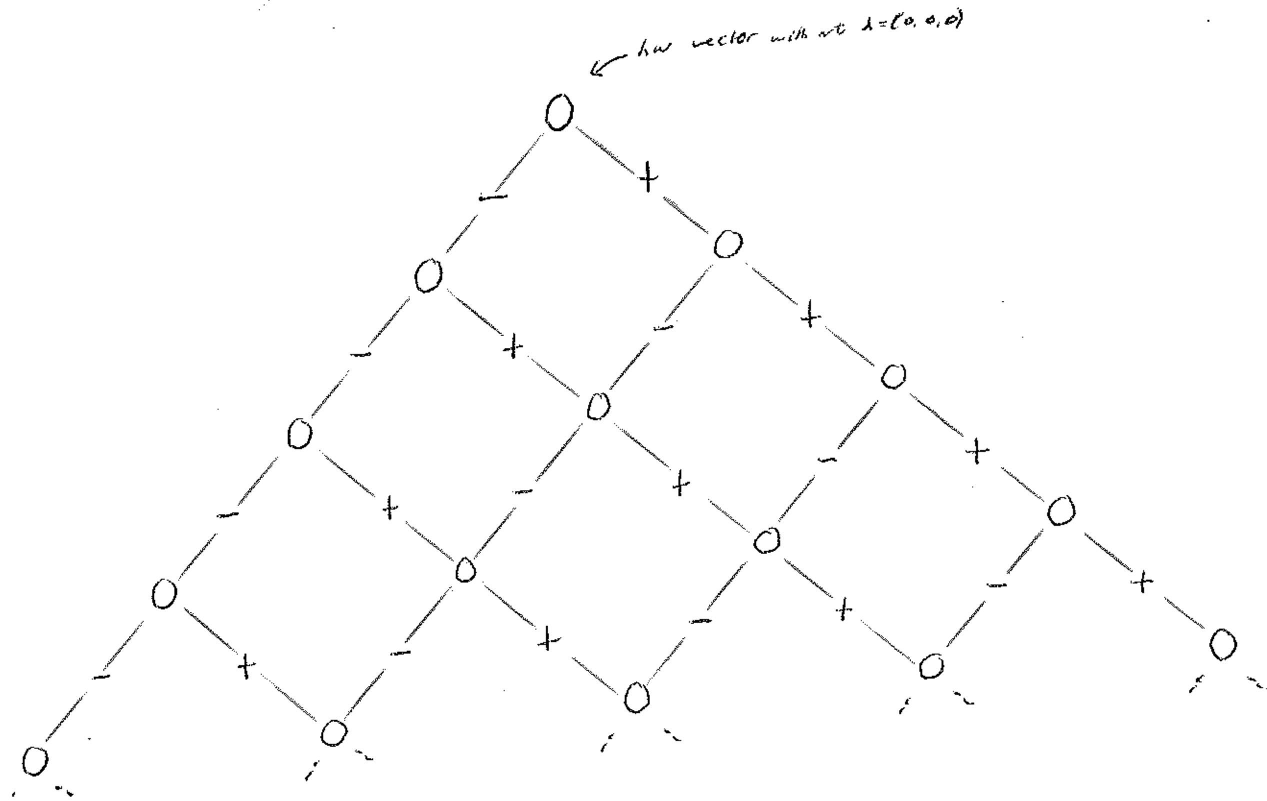
level	-1	0	1
notation	$-$	0	$+$

$\Phi = A_2$, $\quad A = GL(3) \qquad B_\lambda \qquad \lambda = (7, 3, 0)$

To obtain B_∞ from B_λ

- twist to make $\lambda = (0,0,0)$
- replace the rectangle dimensions by ∞, ∞

"subcrystal" of B_∞ induced on levels $-1, 0, 1$ is

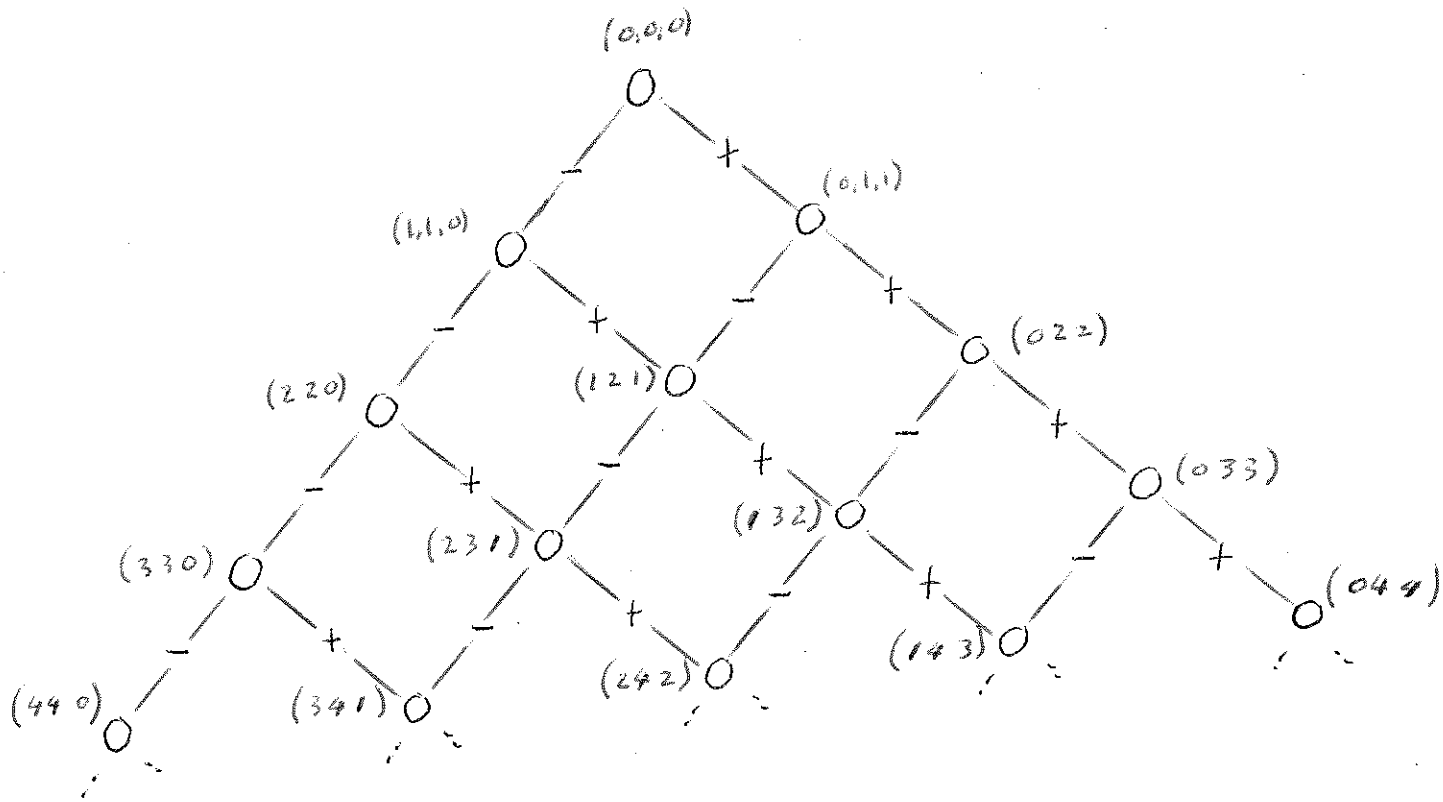


$\Phi = A_2, \quad \Lambda = GL(3) \qquad B_\infty$

Next we identify each element of B_∞ with a 3-tuple

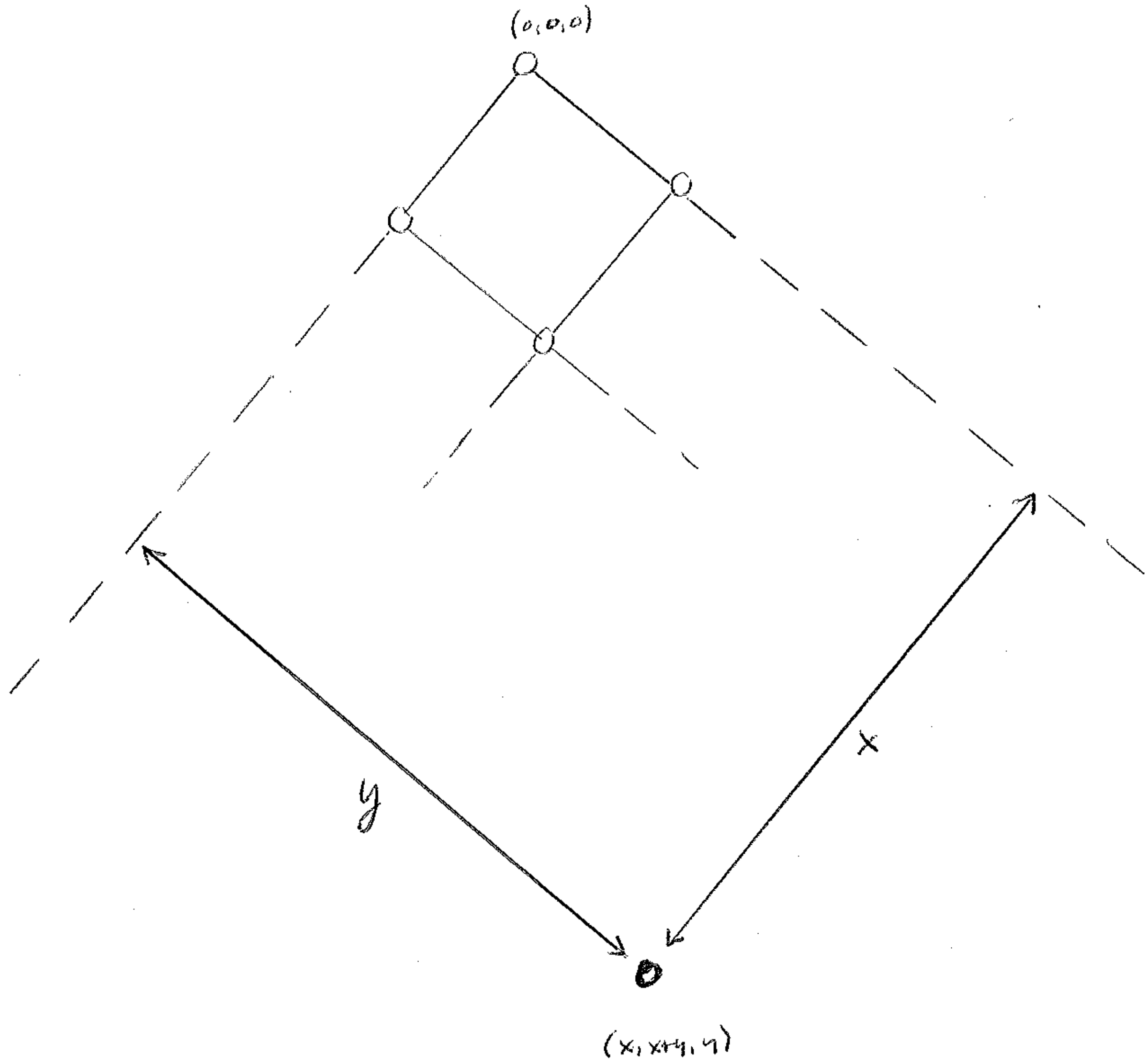
$$(a, b, c) \qquad a, b, c \in \mathbb{N} \qquad \mathbb{N} = \{0, 1, 2, \dots\}$$

We first show the 3-tuple for the elements of level 0:



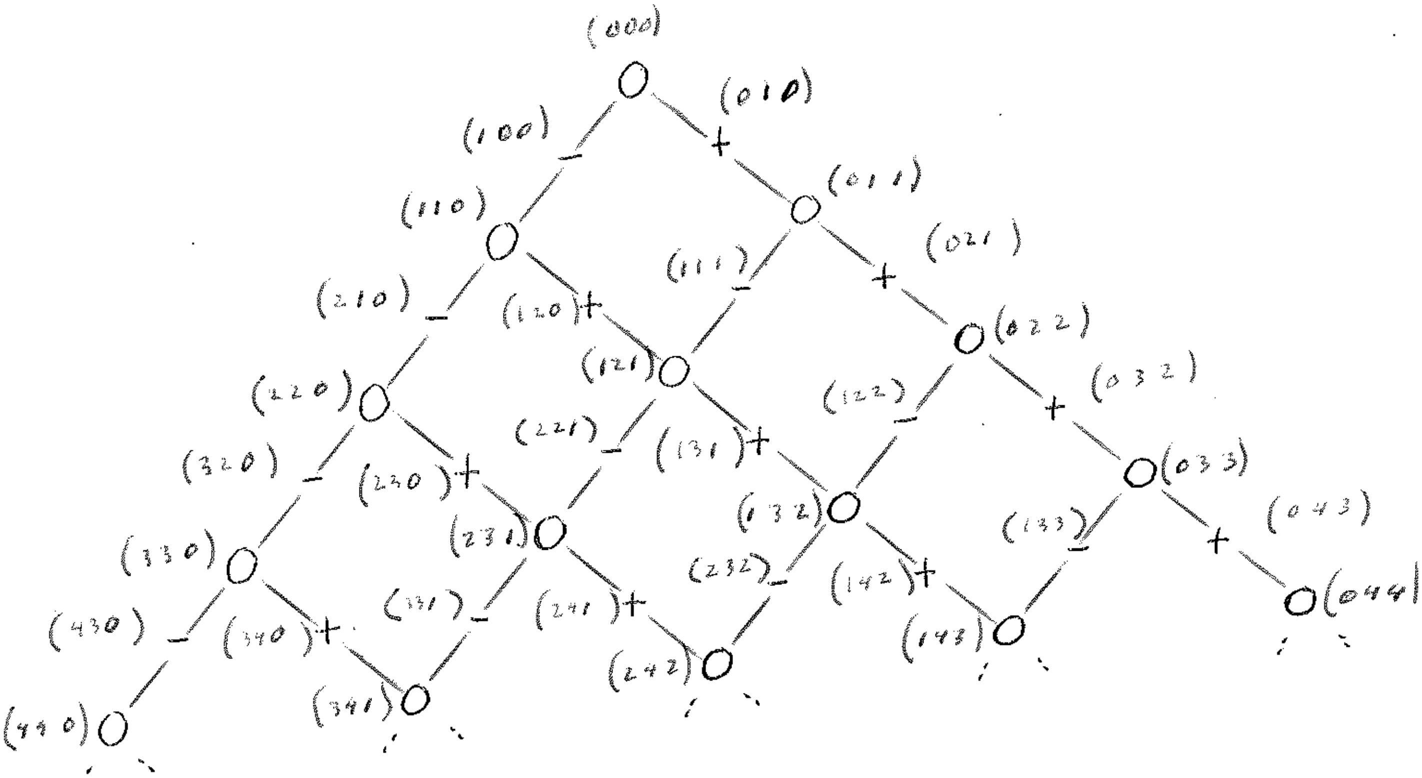
$\Phi = A_2$, $\Lambda = GL(3)$, B_∞

For an element in B_∞ of level 0, the 3-tuple is



$\Phi = A_2$, $\quad \Lambda = GL(3)$, $\quad \mathcal{B}_\infty$

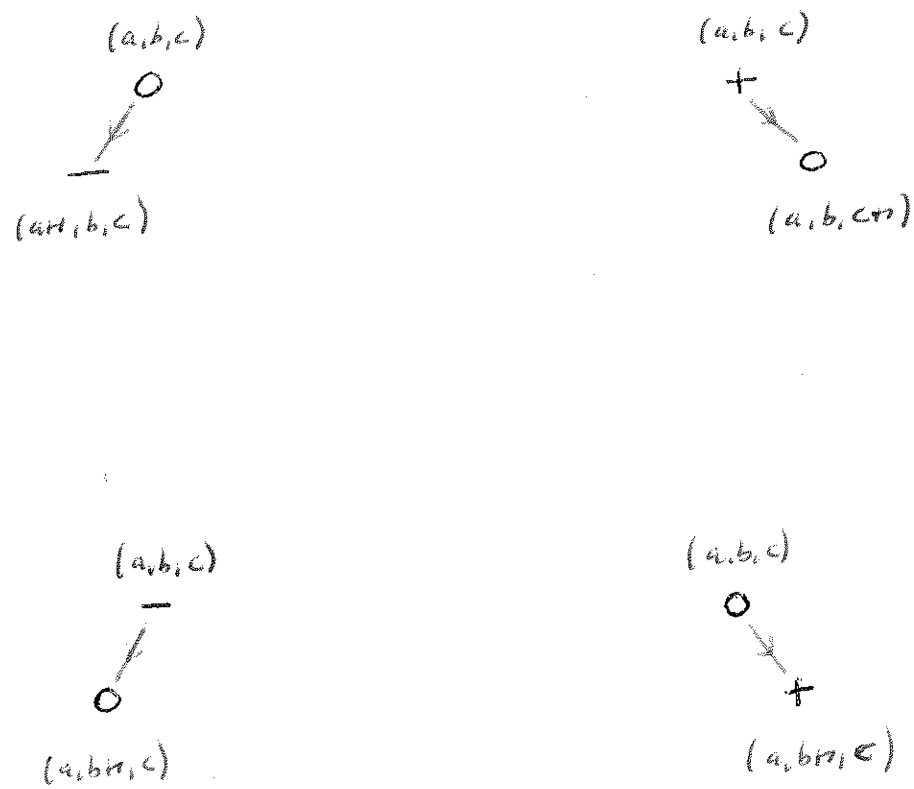
Next we give the 3-tuple for each element of $\mathcal{B}_\infty$ with level $\in \{-1, 0, 1\}$



note (a,b,c) has level $b-a-c$

$\Phi = A_2$, $\Lambda = GL(3)$, B_∞

the pattern in above diagram:



key 

$\Phi = A_2, \quad \Lambda = GL(3), \qquad B_\infty$

The pattern in the above diagram, revisited:

$$(a+1, b, c) \xleftarrow[1]{} (a, b, c) \qquad \text{if} \quad \ell = 0$$

$$(a, b, c+1) \xleftarrow[1]{} (a, b, c) \qquad \text{if} \quad \ell = 1$$

where

$$\ell = b - a - c$$

Also

$$(a, b+1, c) \xleftarrow[2]{} (a, b, c)$$

We now extend above pattern to all of B_∞.

$\Phi = A_2, \quad \Lambda = GL(3), \quad B_\infty$

For all $(a,b,c) \in B_\infty$ we require

$$(a+1,b,c) \xleftarrow{1} (a,b,c) \quad \text{if } \ell \leq 0$$

$$(a,b,c+1) \xleftarrow{1} (a,b,c) \quad \text{if } \ell \geq 1$$

where

$$\ell = b - a - c$$

Also

$$(a,b+1,c) \xleftarrow{2} (a,b,c)$$

We also require

B_∞ is connected and contains $(0,0,0)$

This forces $b \geq c$ for all $(a,b,c) \in B_\infty$ (ex)

We now give the unique sol to above requirements.

Thm For $\Phi = A_2$, $\Lambda = GL(3)$

there exists a crystal B_∞ such that:

$$B_\infty = \left\{ (a, b, c) \;\middle|\; a \geq 0, \quad b \geq c \geq 0 \right\}$$

For $(a, b, c) \in B_\infty$,

$$(a+1, b, c) \xleftarrow{1} (a, b, c) \qquad \text{if } \ell \leq 0$$

$$(a, b, c+1) \xleftarrow{1} (a, b, c) \qquad \text{if } \ell \geq 1$$

where

$$\ell = b - a - c$$

Also

$$(a, b+1, c) \xleftarrow{2} (a, b, c)$$

For $x = (a,b,c) \in B_\infty$ and writing

$$\ell = b - a - c,$$

$$wt(x) = -(a+c)\alpha_1 - b\alpha_2$$

$$\varphi_1(x) = \begin{cases} -a & \text{if } \ell \le 0 \\ -2a+b-c & \text{if } \ell \ge 1 \end{cases}$$

$$\varepsilon_1(x) = \begin{cases} a-b+2c & \text{if } \ell \le 0 \\ c & \text{if } \ell \ge 1 \end{cases}$$

$$\varphi_2(x) = a-b$$

$$\varepsilon_2(x) = b-c$$

Pf We routinely check A1, A2

Fix $x = (a, b, c) \in B_\infty$

$$\langle wt(x), \alpha_1^\vee \rangle \overset{?}{=} \varphi_1(x) - \varepsilon_1(x)$$

$$\| \qquad\qquad\qquad \|$$

$$-\langle (a+c)\alpha_1 + b\alpha_2, \alpha_1^\vee \rangle \qquad -2a-2c-b$$

for all values of l

$$\|$$

$$-2a-2c+b$$

ok

$$\langle wt(x), \alpha_2^\vee \rangle \overset{?}{=} \varphi_2(x) - \varepsilon_2(x)$$

$$\| \qquad\qquad\qquad \|$$

$$a+c-2b$$

$$-\langle (a+c)\alpha_1 + b\alpha_2, \alpha_2^\vee \rangle$$

$$\|$$

$$a+c-2b$$

ok

Rest is clear

$\square$

Prop Fn $\Phi = A_2$, $\Lambda = GL(3)$

the crystal B_∞ satisfies

(i) Fn $x = (a,b,c) \in B_\infty$,

$e_1(x) = \phi$ iff $b \geq a$ and $c = 0$

$e_2(x) = \phi$ iff $b = c$

(ii) B_∞ has unique hw vector $(0,0,0)$

(iii) B_∞ is connected

(iv) Fn $x = (a,b,c) \in B_\infty$,

$\varepsilon_1(x) = \max\{ t \mid t \geq 0, \ e_1^t(x) \neq \phi \}$

$\varepsilon_2(x) = \min\{ t \mid t \geq 0, \ e_2^t(x) \neq \phi \}$

"B_∞ is upper seminormal"

pf (i) use def of $\leftarrow_1$, $\leftarrow_2$

(ii) $x \in B_\infty$ is hw iff $e_1(x) = \phi$ and $e_2(x) = \phi$.

Use this and (i)

(iii) Each connected comp has at least one hw element

(iv) We describe the chain

$$x \xleftarrow[1]{} e_1(x) \xleftarrow[1]{} e_1^2(x) \xleftarrow[1]{} \cdots \qquad \divideontimes$$

write $l = b - a - c$

For $l \geq 1$, $\divideontimes$ is

$$\begin{array}{cc}
 & \text{level} \\
(a,b,0) & l+c \\
\downarrow 1 & \\
(a,b,1) & l+c-1 \\
\downarrow 1 & \vdots \\
\vdots & \\
\downarrow & \\
(a,b,c-1) & l+1 \\
\downarrow 1 & \\
(a,b,c) & l
\end{array}$$

chain length $= c = \varepsilon_1(x)$

For $\ell \le 0$ *15

$$
\begin{array}{cc}
 & \text{level} \\
(b-c, b, 0) & c \\
\downarrow 1 & \\
(b-c, b, 1) & c-1 \\
\downarrow 1 & \\
\vdots & \vdots \\
\downarrow 1 & \\
(b-c, b, c) & 0 \\
\downarrow 1 & \\
(b-c+1, b, c) & -1 \\
\downarrow 1 & \\
\vdots & \vdots \\
\downarrow 1 & \\
(b-c-\ell, b, c) & \ell \\
\parallel & \\
(a, b, c) &
\end{array}
$$

chain length $= c - \ell$

$= a - b + 2c$

$= \varepsilon_1(x)$ ✓

The chain

$$x \xleftarrow{2} e_2(x) \xleftarrow{2} e_2^2(x) \longleftarrow \cdots$$

is

$$\begin{array}{c} (a, c, c) \\ \big\downarrow 2 \\ (a, c+1, c) \\ \big\downarrow 2 \\ \vdots \\ \big\downarrow 2 \\ (a, b, c) \end{array}$$

Chain length $= b - c = \varepsilon_2(x)$ $\square$

$\Phi = A_2, \quad \Lambda = GL(3), \quad B_\infty$

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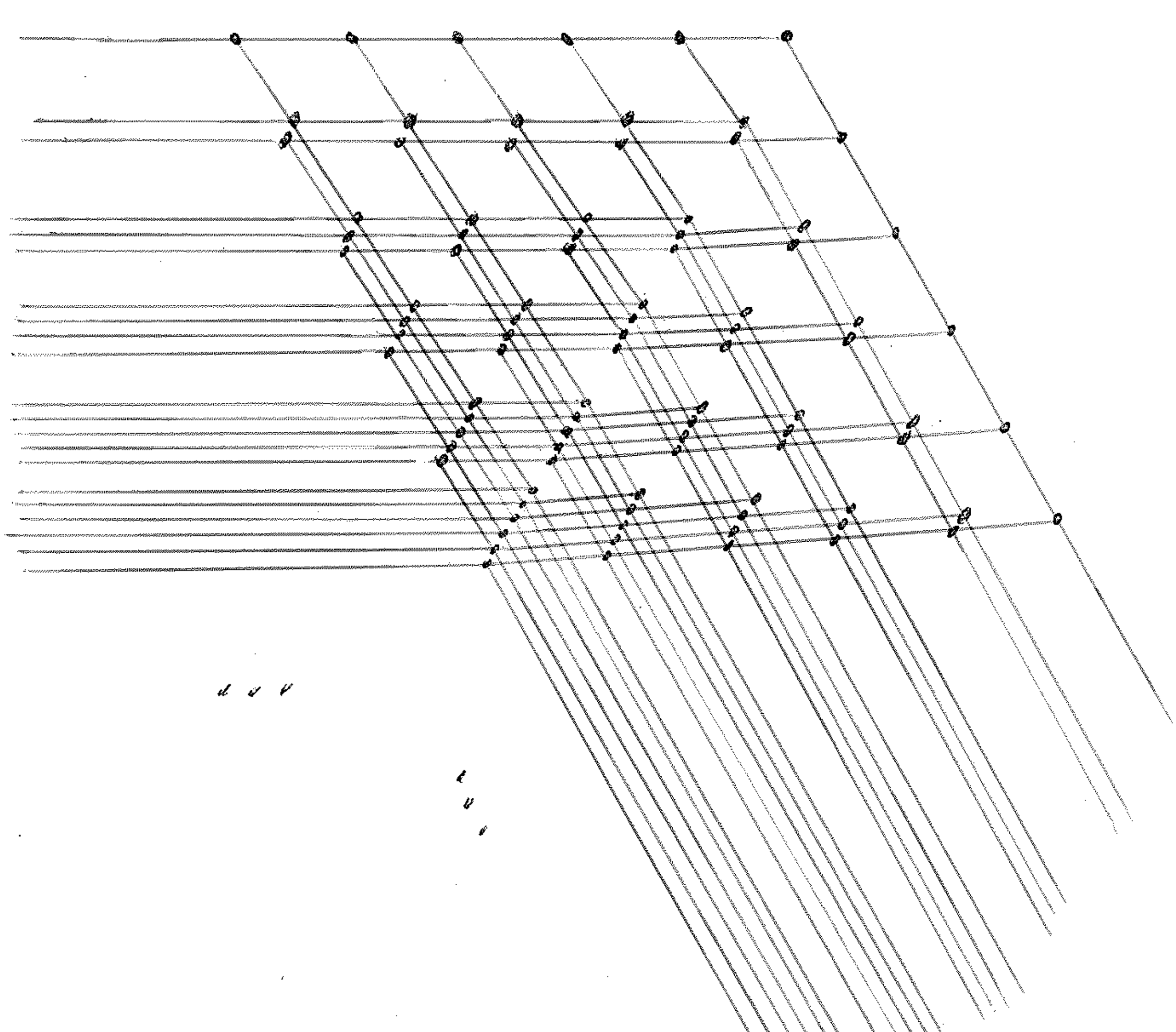
Crystal graph



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