Lecture 28 Wed Nov 6

An involution

For any root system Φ in V with Weyl group W, recall longest element $w_0 \in W$ satisfies

$$w_0(\Phi^{\pm}) = \Phi^{\mp}$$

The map

$$V \xrightarrow[-I]{} V \xrightarrow[w_0]{} V \qquad *$$

is in $O(V)$ and sends $\Phi^{\pm} \to \Phi^{\pm}$

So $*$ is a diagram aut. or ident.

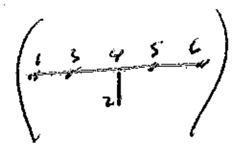
$*$ permutes $\{\alpha_i\}_{i \in I}$

$*$ sends

$$\alpha_i \longrightarrow \alpha_{i'} \qquad i \in I$$

$*^2 = \mathrm{id}$ so

$$(i')' = i \qquad i \in I$$

Φ	$-w_0$	w_0	i'	
A_r	aut	$e_i \to e_{n+1-i}$	$n-i$	$(n=r+1)$
B_r	id	$e_i \to -e_i$	i	
C_r	id	$e_i \to -e_i$	i	
D_r r even	id	$e_i \to -e_i$	i	
r odd	aut	$e_i \to -e_i \ (1 \le i \le r-1)$, $e_r \to e_r$	$\begin{cases} i & \text{if } 1 \le i \le r-2 \\ r & i = r-1 \\ r-1 & i = r \end{cases}$	
E_6	aut	$e_1 \leftrightarrow e_8$, $e_2 \leftrightarrow e_7$, $e_3 \leftrightarrow e_6$, $e_4 \leftrightarrow e_5$	$1'=6$, $2'=2$, $3'=5$, $4'=4$, $5'=3$, $6'=1$	
E_7	id	$e_i \to -e_i$	i	
E_8	id	$e_i \to -e_i$	i	
F_4	id	$e_i \to -e_i$	i	
G_2	id	$e_i \to -e_i$	i	

An involution for crystals

Consider standard crystal for $A_r, B_r, C_r, D_r, \ldots$

We describe "left-right symmetry" in crystal graph

$\Phi = A_r$:

$$B: \quad \boxed{r+1} \xleftarrow{r} \boxed{r} \xleftarrow{r-1} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

$$wt: \quad e_{r+1} \qquad e_r \qquad \cdots \qquad e_2 \qquad e_1$$

Define a new crystal with vertex set B :

$$B^{new}: \quad \boxed{r+1} \xrightarrow{1} \boxed{r} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{2} \xrightarrow{r} \boxed{1}$$

$$wt^{new}: \quad e_1 \qquad e_2 \qquad \cdots \qquad e_r \qquad e_{r+1}$$

B^{new} is obtained from B by

- change direction of each arrow
- replace each edge label i by i'
- For $b \in B$,

$$wt^{new}(b) = w_0(wt(b))$$

Obs B, B^{new} are isomorphic

$$\Phi = B_r$$

$$B: \quad \boxed{\bar 1} \xleftarrow{1} \boxed{\bar 2} \xleftarrow{2} \cdots \xleftarrow{r-1} \boxed{\bar r} \xleftarrow{r} \boxed{0} \xleftarrow{r} \boxed{r} \xleftarrow{r-1} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

$$\mathrm{wt}: \quad -e_1 \qquad -e_2 \qquad -e_r \qquad 0 \qquad e_r \qquad e_2 \qquad e_1$$

$$B^{new} \quad \boxed{\bar 1} \xrightarrow{1} \boxed{\bar 2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{\bar r} \xrightarrow{r} \boxed{0} \xrightarrow{r} \boxed{r} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{2} \xrightarrow{1} \boxed{1}$$

$$\mathrm{wt}^{new} \quad e_1 \qquad e_2 \qquad e_r \qquad 0 \qquad -e_r \qquad -e_2 \qquad -e_1$$

crystals B, B^{new} iso ✓

$\Phi = C_r$

$$B \qquad \boxed{\bar 1} \xleftarrow{1} \boxed{\bar 2} \xleftarrow{2} \cdots \xleftarrow{r-1} \boxed{\bar r} \xleftarrow{r} \boxed{r} \xleftarrow{r-1} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

$$\mathrm{wt} \qquad -e_1 \quad -e_2 \quad \cdots \quad -e_r \quad e_r \quad \cdots \quad e_2 \quad e_1$$

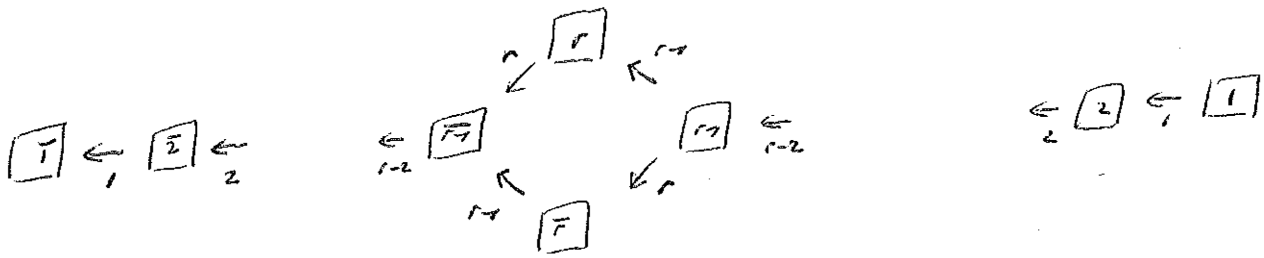
$$B^{new} \qquad \boxed{\bar 1} \xrightarrow{1} \boxed{\bar 2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{\bar r} \xrightarrow{r} \boxed{r} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{2} \xrightarrow{1} \boxed{1}$$

$$\mathrm{wt}^{new} \qquad e_1 \quad e_2 \quad \cdots \quad e_r \quad -e_r \quad \cdots \quad -e_2 \quad -e_1$$

crystals B, B^{new} iso

$\Phi = D_r$

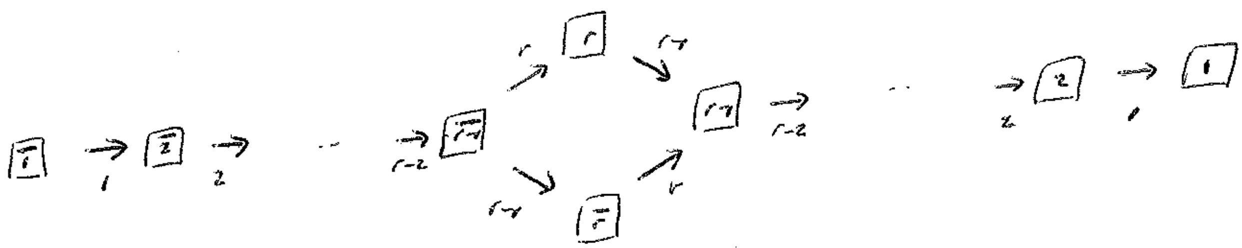
B:



wt $-e_1$ $-e_2$ $-e_{r-1}$ e_r $-e_r$ e_{r-1} e_2 e_1

r even:

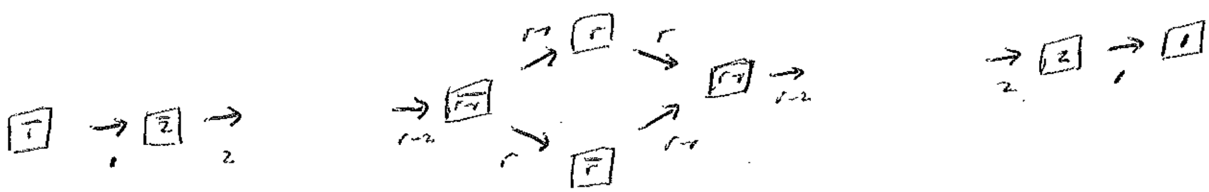
B^{new}:



wt^{new} e_1 e_2 … e_{r-1} $-e_r$ e_r $-e_{r-1}$ … $-e_2$ $-e_1$

r odd:

B^{new}:



wt^{new}: e_1 e_2 e_{r-1} e_r $-e_r$ $-e_{r-1}$ $-e_2$ $-e_1$

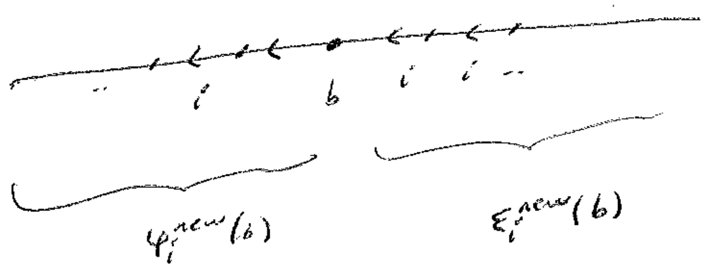
crystals B, B^{new} iso

Prop. Given seminormal crystal B with rt data Φ, Λ then $\exists$ crystal B^{new} for Φ, Λ obtained from B via ☆ above. Moreover B^{new} is seminormal and $(B^{new})^{new} = B$

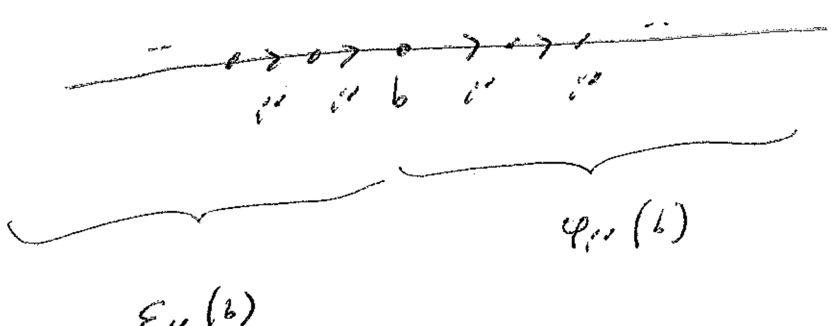
pf For $b \in B$ and $i \in I$ check

$$\langle wt^{new}(b), \alpha_i^\vee \rangle \overset{?}{=} \varphi_i^{new}(b) - \varepsilon_i^{new}(b) \qquad *$$

new



orig



So

$\varphi_i^{new}(b) = \varepsilon_{i'}(b)$, $\quad \varepsilon_i^{new}(b) = \varphi_{i'}(b)$

In $*$,

$$LHS = \langle w_0(wt(b)), \alpha_i^\vee \rangle$$

$$= \langle wt(b), \underset{\substack{\| \\ -(\alpha_{i^*})^\vee}}{w_0(\alpha_i^\vee)} \rangle$$

$$= -\langle wt(b), (\alpha_{i^*})^\vee \rangle$$

$$RHS = \varepsilon_{i^*}(b) - \varphi_{i^*}(b)$$

$$= -\langle wt(b), (\alpha_{i^*})^\vee \rangle$$

$*$ holds ✓ $\square$

Rest is clear.

With above notation, it turns out TFAE

(i) B is normal

(ii) B^{new} is normal

Thm Given normal crystal B with rt data Φ, Λ

Then the crystals B, B^{new} are isomorphic.

pf Normal crystal B has unique hw λ

Normal crystal B^{new} has unique hw λ^{new}

Suf to show $\lambda = \lambda^{new}$

Since λ is a wt for B,

$w_0(\lambda)$ is a wt for B $\quad *$

Since λ is hw for B

$\lambda + \alpha_i$ not a wt for B $\forall i \in I$

So

$$\underbrace{w_0(\lambda + \alpha_i)}_{= w_0(\lambda) + \underbrace{w_0(\alpha_i)}_{= -\alpha_{i'}}} \text{ not a wt for } B \quad \forall i \in I$$

$w_0(\lambda) - \alpha_j$ not a wt for B $\forall j \in I$. $\quad **$

By $*$,

$\exists\ x \in B$ st $wt(x) = w_0(\lambda)$

Obs

$$wt^{new}(x) = w_i(\lambda) = \lambda$$

By $**$,

$$f_j(x) = \phi \qquad \forall j \in I$$

$$\begin{array}{l} \| \\ e_{j'}^{new}(x) \end{array}$$

$$e_i^{new}(x) = \phi \qquad \forall i \in I$$

So

x is hw in B^{new}

thus

λ is hw for B^{new}

So $\lambda^{new} = \lambda$

$\square$

DEF Given a normal crystal B with rt data Φ, Λ.

The <u>crystal involution</u> for B is the (unique) crystal

iso

$$S : B \to B^{new}$$

By constr,

$$S^2 = id$$

and for $x \in B$

$$wt(S(x)) = w_0(wt(x))$$

$$\varphi_i(S(x)) = \varepsilon_{i'}(x)$$

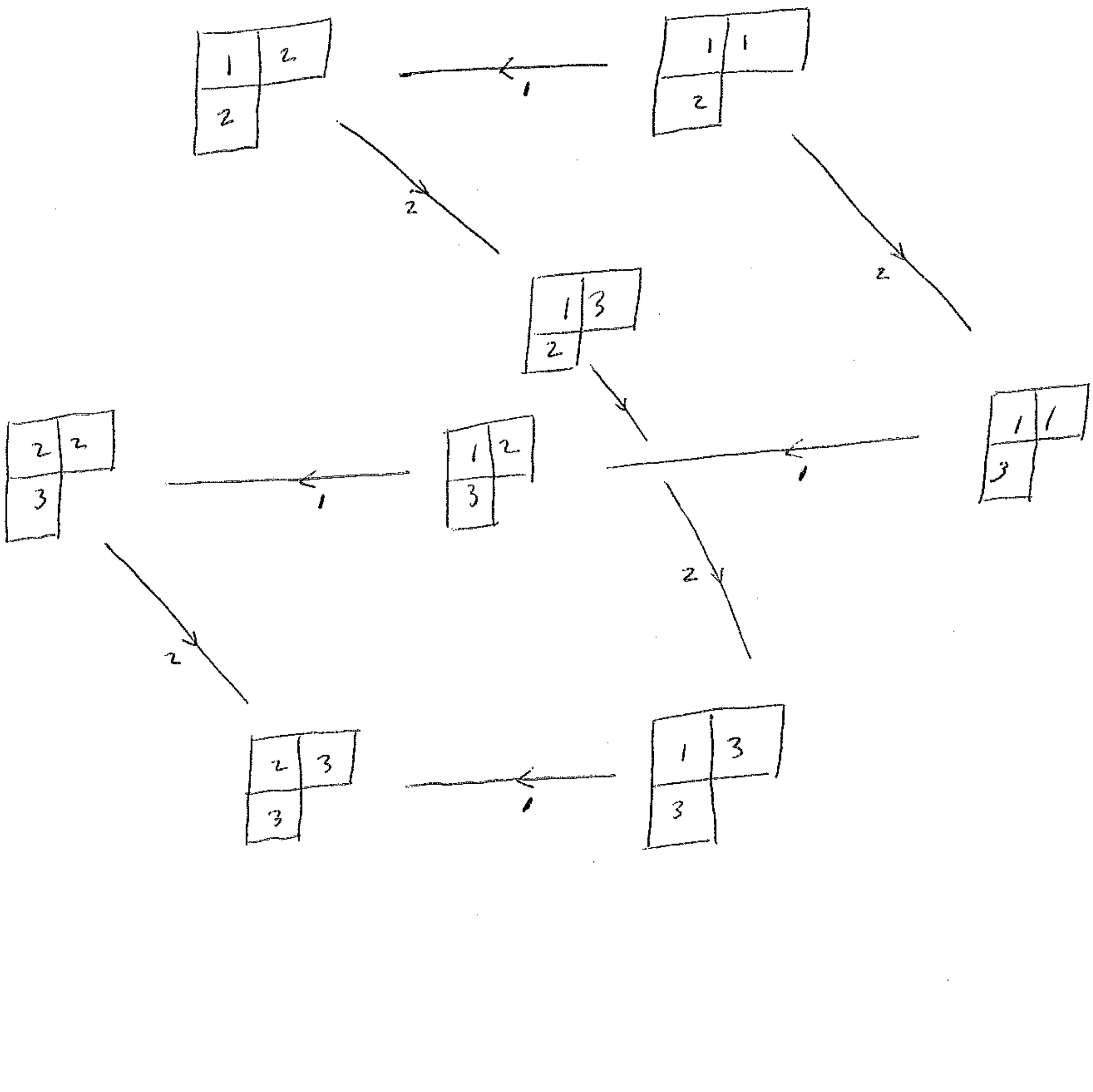
$$\varepsilon_i(S(x)) = \varphi_{i'}(x) \qquad i \in I$$

$$f_i(S(x)) = e_{i'}(x)$$

$$e_i(S(x)) = f_{i'}(x)$$

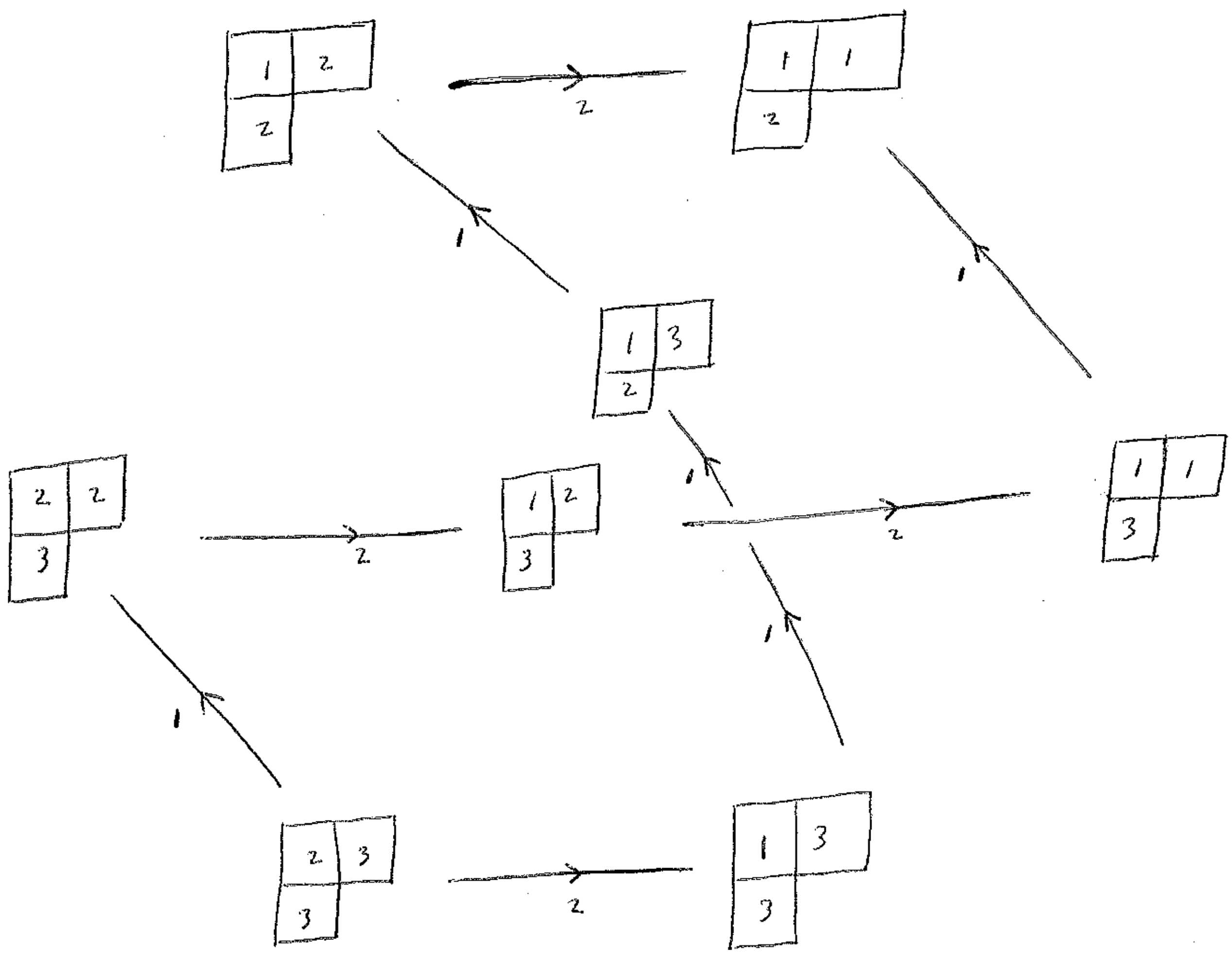
Ex $\quad \Phi = A_2, \quad \Lambda = GL(3) \quad \lambda = (2,1)$

Showing $\quad B = B_\lambda$



$\Phi = A_2, \quad \Lambda = GL(3), \quad \lambda = (2,1) \quad B = B_\lambda$

showing B^{new} (prelim)



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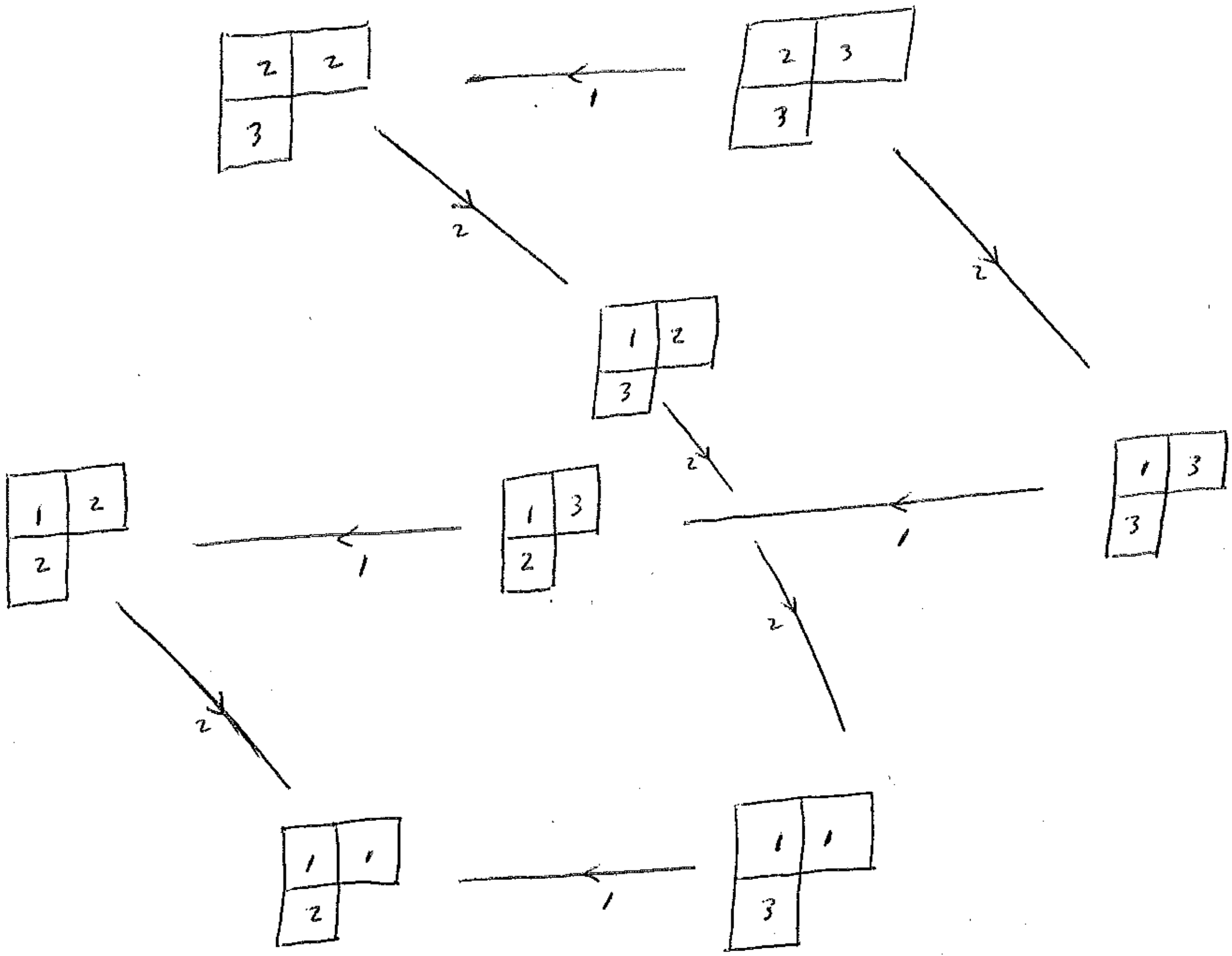
$\Phi = A_2, \quad \Lambda = GL(3), \quad \lambda = (2,1) \quad \mathcal{B} = \mathcal{B}_\lambda$

showing $\mathcal{B}^{new}$



$\Phi = A_2, \quad \Lambda = GL(3) \qquad \lambda = (2,1) \qquad B = B_\lambda$

crystal involution S sends

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array} \longleftrightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} \longleftrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1} \end{array} \longleftrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \longleftrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$$