

LEM Given irred root system Φ
and wt lattice Λ .

Given adjoint crystal B for Φ, Λ .

Then B is connected. Moreover B has
unique hw vects as shown below

Φ	hw vects
A_r	$e_1 - e_{r+1} = x_1 + \dots + x_r = (\bar{w}_1 + \bar{w}_r \text{ for } \Lambda = SL(n, \mathbb{Z}))$
B_r	$e_1 + e_2 = x_1 + 2x_2 + \dots + 2x_r = \bar{w}_2$
C_r	$2e_1 = 2x_1 + 2x_2 + \dots + 2x_{r-1} + x_r = 2\bar{w}_1$
D_r	$e_1 + e_2 = x_1 + 2x_2 + \dots + 2x_{r-2} + x_{r-1} + x_r = \bar{w}_2$
E_6	\bar{w}_2
E_7	\bar{w}_1
E_8	\bar{w}_8
F_4	\bar{w}_1
G_2	\bar{w}_2

pf In each case one checks given vector

is unique root $\alpha \in \Phi$ st

$$\alpha + \alpha_i \notin \Phi \quad \forall i \in I$$

"highest root"

So α is unique hw vector.

Crystal B is connected since each connected component

has at least one hw vector. □

Recall

Given a root system Φ from classification:

$$B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2$$

$$\Lambda = \Lambda_{sc}$$

Desire to define a fundamental crystal $B_{\bar{w}_k}$ for $k \in I$

For Φ simply laced,

$B_{\bar{w}_k}$ should be stembridge

For Φ not simply laced,

$B_{\bar{w}_k}$ should be a connected component of a tensor product of virtual crystals

Cases

B_r, C_r, D_r done ✓

Now consider

$$E_6, E_7, E_8, F_4, G_2$$

(sketch)

$$\Phi = E_6:$$

wt's \bar{w}_1, \bar{w}_6 are miniscule

Corresp miniscule crystals $B_{\bar{w}_1}, B_{\bar{w}_6}$ are stembridge

Define crystal

$$\begin{array}{lcl}
 B_{\bar{w}_2} = \text{unique connected component of } B_{\bar{w}_1} \otimes B_{\bar{w}_6} & \text{with hw} & \bar{w}_2 \\
 B_{\bar{w}_3} = \dots & B_{\bar{w}_5} \otimes B_{\bar{w}_1} & \dots \bar{w}_3 \\
 B_{\bar{w}_4} = \dots & B_{\bar{w}_1} \otimes B_{\bar{w}_1} \otimes B_{\bar{w}_1} & \dots \bar{w}_4 \\
 B_{\bar{w}_5} = \dots & B_{\bar{w}_6} \otimes B_{\bar{w}_6} & \dots \bar{w}_5
 \end{array}$$

$\Phi = E_7$:

wt \bar{w}_4 is minuscule
 Corresp minuscule crystal $B_{\bar{w}_7}$ is Skewbridge

Define crystal

$$\begin{array}{lcl}
 B_{\bar{w}_1} = \text{unique connected component of } B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \text{with hw} & \bar{w}_1 \\
 B_{\bar{w}_2} = \dots & B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \dots \bar{w}_2 \\
 B_{\bar{w}_3} = \dots & B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \dots \bar{w}_3 \\
 B_{\bar{w}_4} = \dots & B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \dots \bar{w}_4 \\
 B_{\bar{w}_5} = \dots & B_{\bar{w}_7} \otimes B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \dots \bar{w}_5 \\
 B_{\bar{w}_6} = \dots & B_{\bar{w}_7} \otimes B_{\bar{w}_7} & \dots \bar{w}_6
 \end{array}$$

$$\overline{\Phi} = E_8$$

Adjoint crystal is Stembridge and has hw \overline{w}_8

Call this crystal $B\overline{w}_8$

For $1 \leq k \leq 7$, $B\overline{w}_k$ is defined to be a certain specific connected component of

$$(B\overline{w}_8)^{\otimes l} \quad l \in \{2, 3, 4, 5\}$$

that has hw \overline{w}_k

(see table 5.5 in text)

$$\Phi = F_4$$

Using the virtual pairing F_4, E_6

a virtual crystal for F_4 is constructed with hw $\bar{\omega}_4$

Call this crystal $B\bar{\omega}_4$

Define crystal

$B\bar{\omega}_1 =$ unique connected component of $B\bar{\omega}_4 \otimes B\bar{\omega}_4$ with hw $\bar{\omega}_1$

$B\bar{\omega}_2 =$ $\bar{\omega}_2$

$B\bar{\omega}_3 =$ $\bar{\omega}_3$

Turns out $B\bar{\omega}_1$ is iso to the adjoint crystal for F_4

$$\Phi = G_2$$

Using the virtual pairing G_2, D_4

a virtual crystal for G_2 is constructed with hw $\bar{\omega}_1$

Call this crystal $B\bar{\omega}_1$

Define crystal

$B\bar{\omega}_2 =$ unique connected component of $B\bar{\omega}_1 \otimes B\bar{\omega}_1$ with hw $\bar{\omega}_2$

turns out $B\bar{\omega}_2$ is iso to adjoint crystal for G_2

For $\overline{\mathbb{Q}}$ among

$B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$

$$\Lambda = \Lambda_{sc}$$

we have defined a crystal $B_{\overline{w}_k}$ for $k \in I$

Next, for $\lambda \in \Lambda^+$ we define a crystal B_λ .

write

$$\lambda = \sum_{k \in I} a_k \overline{w}_k \quad a_k \in \mathbb{N}$$

The crystal

$$\bigotimes_{k \in I} (B_{\overline{w}_k})^{\otimes a_k}$$

contains a connected component (denoted B_λ) with

hw λ and hw vector

$$\bigotimes_{k \in I} (u_k)^{\otimes a_k}$$

$$(u_k = \text{hw vector for } B_{\overline{w}_k})$$

We just defined crystal B_λ for $\lambda \in \Lambda^+$

For Φ among X
 For any wt lattice Λ from classification (not nec Λ_{ac})

Define B_λ for $\lambda \in \Lambda^+$:

view $\lambda \in \Lambda \subseteq \Lambda_{ac}$

By constr $\lambda \in \Lambda_{ac}^+$

So we have crystal

B_λ for Φ, Λ_{ac}

wt function

$wt: B_\lambda \rightarrow \Lambda_{ac}$

has image $\subseteq \Lambda$

So we may view B_λ as crystal for Φ, Λ .

Back to $\mathbb{F} = A_n$, $\Lambda = GL(n, \mathbb{R})$, $n = \text{rank}$

For $\lambda \in \Lambda$ write

$$\lambda = \sum_{i=1}^n \lambda_i e_i \quad \lambda_i \in \mathbb{Z}$$

$\lambda \in \Lambda^+$ iff $\lambda_1 \geq \dots \geq \lambda_n$
 ↑
 could be neg

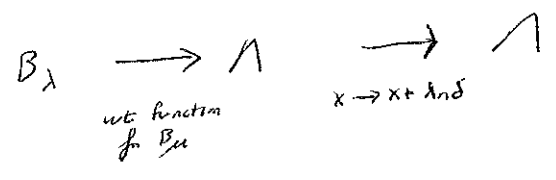
Assume $\lambda \in \Lambda^+$ and $\lambda_n < 0$. Define B_λ

Write

$$\begin{aligned} \mu &= \lambda - \lambda_n \delta & \delta &= e_1 + e_2 + \dots + e_n \\ &= (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \dots, 0) \\ &= \text{partition.} \end{aligned}$$

\exists crystal B_μ for Λ

Define B_λ to be the twist of B_μ with wt function



The crystal B_λ has unique h.w vector, h.w λ .

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$n = r + 1$

For $\Phi = A_r$, $\Lambda = GL(r+1)$
 $\Lambda' = SL(r+1)$

next goal:

For $\lambda \in (\Lambda')^+$ define B_λ .

Write

$$\lambda = \sum_{i=1}^n \lambda_i e_i$$

$$\sum_{i=1}^n \lambda_i = 0$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$$\lambda_i - \lambda_j \in \mathbb{Z}$$

Define

$$\begin{aligned} \mu &= \lambda - \lambda_n \delta \\ &= (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \dots, 0) \\ &= \text{partition in } \Lambda^+ \end{aligned}$$

$$\delta = e_1 + \dots + e_n$$

\exists crystal $B = B_\mu$ for Λ

isogeny

$$m: \Lambda \rightarrow \Lambda'$$

$$e_i \rightarrow e_i - \frac{\delta}{n}$$

sends

$$\mu \rightarrow \lambda$$

B becomes a crystal for Λ' with wt function

$$B \xrightarrow{\text{wt}} \Lambda \xrightarrow{m} \Lambda'$$

this gives crystal B_λ for Λ'

B_λ has unique h.w vector, h.w λ ,

For all Φ among

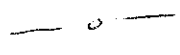
$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$

and Λ from classification

We defined crystal B_λ for $\lambda \in \Lambda^+$

Call a crystal normal whenever it is iso to some B_λ above.

Note: In the text, the def of normal crystal is a lot different, because they assume λ is any wt lattice for Φ , not nec from classification



- By constn each normal crystal B_λ has unique hw elements, with hw λ
- turns out: for normal crystals B_λ, B_μ with same root data, each connected component of $B_\lambda @ B_\mu$ is normal