LEM Given irred root system Φ and wt lattice Λ.
Given adjoint crystal $\mathcal{B}$ for Φ, Λ.
Then $\mathcal{B}$ is connected. Moreover $\mathcal{B}$ has unique hw vector as shown below

Φ	hw vector
A_r	$e_1 - e_{r+1} = \alpha_1 + \cdots + \alpha_r$ $(= \overline{\omega}_1 + \overline{\omega}_r$ for $\Lambda = SL(r+1))$
B_r	$e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r = \overline{\omega}_2$
C_r	$2e_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r = 2\overline{\omega}_1$
D_r	$e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r = \overline{\omega}_2$
E_6	$\overline{\omega}_2$
E_7	$\overline{\omega}_1$
E_8	$\overline{\omega}_8$
F_4	$\overline{\omega}_1$
G_2	$\overline{\omega}_2$

pf In each case one checks given vector

is unique root $\alpha \in \Phi$ st

$$\alpha + \alpha_i \notin \Phi \quad \text{fr } i \in I$$ "highest root"

So α is unique hw vector.

Crystal B is connected since each connected component

has at least one hw vector. $\square$

Recall

Given a root system Φ from classification:

$$B_r,\ C_r,\ D_r,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

$\Lambda = \Lambda_{sc}$

Desire to define a fundamental crystal B_{ϖ_k} for $k \in I$

For Φ simply laced,

B_{ϖ_k} should be Stembridge

For Φ not simply laced,

B_{ϖ_k} should be a connected component of a tensor product of virtual crystals

Cases $B_r,\ C_r,\ D_r$ done ✓

Now consider

$$E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

(sketch)

$\Phi = E_6$:

wts $\varpi_1,\ \varpi_6$ are miniscule

Corresp miniscule crystals $B_{\varpi_1},\ B_{\varpi_6}$ are Stembridge

Define crystal

$B_{\bar\omega_2}$ = unique connected component of $B_{\bar\omega_1} \otimes B_{\bar\omega_6}$ with hw $\bar\omega_2$

$B_{\bar\omega_3}$ = ... $B_{\bar\omega_1} \otimes B_{\bar\omega_1}$ " $\bar\omega_3$

$B_{\bar\omega_4}$ = ... $B_{\bar\omega_1} \otimes B_{\bar\omega_1} \otimes B_{\bar\omega_1}$ " $\bar\omega_4$

$B_{\bar\omega_5}$ = ... $B_{\bar\omega_6} \otimes B_{\bar\omega_6}$ " $\bar\omega_5$

$\Phi = E_7$:

wt $\bar\omega_7$ is miniscule

Corresp miniscule crystal $B_{\bar\omega_7}$ is Stembridge

Define crystal

$B_{\bar\omega_1}$ = unique connected component of $B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ with hw $\bar\omega_1$

$B_{\bar\omega_2}$ = .. $B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ .. $\bar\omega_2$

$B_{\bar\omega_3}$ = -- $B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ $\bar\omega_3$

$B_{\bar\omega_4}$ = -- $B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ $\bar\omega_4$

$B_{\bar\omega_5}$ = -- $B_{\bar\omega_7} \otimes B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ $\bar\omega_5$

$B_{\bar\omega_6}$ = -- $B_{\bar\omega_7} \otimes B_{\bar\omega_7}$ $\bar\omega_6$

$\Phi = E_8$

Adjoint crystal is Stembridge and has hw ϖ_8

Call this crystal $B\varpi_8$

For $1 \le k \le 7$, $B\varpi_k$ is defined to be a certain specific connected component of

$$(B\varpi_8)^{\otimes \ell} \qquad \ell \in \{2, 3, 4, 5\}$$

that has hw ϖ_k

(See table 5.5 in text)

$\Phi = F_4$

Using the virtual pairing F_4, E_6

a virtual crystal for F_4 is constructed with hw ϖ_4

Call this crystal $B\varpi_4$

Define crystal

$B\varpi_1$ = unique connected component of $B\varpi_4 \otimes B\varpi_4$ with hw ϖ_1

$B\varpi_2$ = " " ϖ_2

$B\varpi_3$ = " " ϖ_3

Turns out $B\varpi_1$ is iso to the adjoint crystal for F_4

$\Phi = G_2$

Using the virtual pairing G_2, D_4

a virtual crystal for G_2 is constructed with hw ϖ_1

Call this crystal $B\varpi_1$

Define crystal

$B\varpi_2$ = unique connected component of $B\varpi_1 \otimes B\varpi_1$ with hw ϖ_2

turns out $B\varpi_2$ is iso to adjoint crystal for G_2

For Φ among

B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2 *

$\Lambda = \Lambda_{sc}$

we have defined a crystal B_{ϖ_k} for $k \in I$

Next, for $\lambda \in \Lambda^+$ we define a crystal B_λ.

write

$$\lambda = \sum_{k \in I} a_k \varpi_k \qquad a_k \in \mathbb{N}$$

The crystal

$$\bigotimes_{k \in I} \left(B_{\varpi_k} \right)^{\otimes a_k}$$

contains a connected component (denoted B_λ) with

hw λ and hw vector

$$\bigotimes_{k \in I} (u_k)^{\otimes a_k} \qquad \left(u_k = \text{hw vector for } B_{\varpi_k} \right)$$

We just defined crystal B_λ for $\lambda \in \Lambda^+$

For Φ among X

For any wt lattice Λ from classification (not nec Λ_{sc})

Define B_λ for $\lambda \in \Lambda^+$:

View $\lambda \in \Lambda \subseteq \Lambda_{sc}$

By constr $\lambda \in \Lambda_{sc}^+$

So we have crystal

B_λ for Φ, Λ_{sc}

wt function

$wt: B_\lambda \to \Lambda_{sc}$

has image $\subseteq \Lambda$

So we may view B_λ as crystal for Φ, Λ.

Back to $\Phi = A_{r}$, $\Lambda = GL(r+1)$ $n = r+1$

For $\lambda \in \Lambda$ write

$$\lambda = \sum_{i=1}^{n} \lambda_i e_i \qquad \lambda_i \in \mathbb{Z}$$

$\lambda \in \Lambda^+$ iff $\lambda_1 \geq \cdots \geq \lambda_n$ ← could be neg

Assume $\lambda \in \Lambda^+$ and $\lambda_n < 0$. Define B_λ

Write

$$\mu = \lambda - \lambda_n \delta \qquad \delta = e_1 + e_2 + \cdots + e_n$$
$$= (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, 0)$$
$$= \text{partition.}$$

$\exists$ crystal B_μ for Λ

Define B_λ to be the twist of B_μ with wt function

$$B_\lambda \xrightarrow[\substack{\text{wt function} \\ \text{for } B_\mu}]{} \Lambda \xrightarrow[x \to x + \lambda_n \delta]{} \Lambda$$

The crystal B_λ has unique hw vector, hw λ.

For $\Phi = A_r$, $\Lambda = GL(r+1)$, $\Lambda' = SL(r+1)$ $n = r+1$

next goal:
For $\lambda \in (\Lambda')^+$ define B_λ.

Write $\lambda = \sum_{i=1}^{n} \lambda_i e_i$ $\sum_{i=1}^{n} \lambda_i = 0$

$\lambda_i - \lambda_j \in \mathbb{Z}$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

Define $\mu = \lambda - \lambda_n \delta$ $\delta = e_1 + \dots + e_n$

$= (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \dots, 0)$

$=$ partition in Λ^+

$\exists$ crystal $B = B_\mu$ for Λ

isogeny

$$m : \begin{array}{ccc} \Lambda & \longrightarrow & \Lambda' \\ e_i & \longrightarrow & e_i - \frac{\delta}{n} \end{array}$$

sends $\mu \longrightarrow \lambda$

B becomes a crystal for Λ' with wt function

$$B \xrightarrow[wt]{} \Lambda \xrightarrow[m]{} \Lambda'$$

This gives crystal B_λ for Λ'

B_λ has unique hw vector, hw λ, For all Φ among

$A_r, B_r, C_r, D_r, \quad E_6, E_7, E_8, \quad F_4, \quad G_2$

and Λ from classification

We defined crystal B_λ for $\lambda \in \Lambda^+$

Call a crystal <u>normal</u> whenever it is iso to some B_λ above.

Note: In the text, the def of normal crystal is a lot different, because they assume λ is any wt lattice for Φ, not nec from classification

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- By constr each normal crystal B_λ has unique hw elements, with hw λ
- turns out: for normal crystals B_λ, B_μ with same root data, each connected component of $B_\lambda \otimes B_\mu$ is normal