Lecture 26, Friday November 1

Thm For a root system Φ and wt lattice Λ

Given crystal $\mathcal{B}$ for Φ, Λ

TFAE:

(i) $\mathcal{B}$ is miniscule;

(ii) $\mathcal{B}$ is feasibly miniscule and

$$wt(x) \neq wt(y) \quad \text{if } x \neq y \qquad x, y \in \mathcal{B}$$

pf (i) → (ii) ✓

(ii) → (i) The map

$$wt: \quad \mathcal{B} \longrightarrow \Omega$$

is a bijective strict crystal morphism, and hence a crystal isomorphism.

Ω is miniscule so $\mathcal{B}$ is miniscule. $\square$

Thm For a root system Φ and wt lattice Λ.
Given a crystal $\mathcal{B}$ for Φ, Λ

TFAE:

(i) $\mathcal{B}$ is miniscule

(ii) $\mathcal{B}$ is feasibly miniscule and has a unique hw vector

pf (i) → (ii)

$\mathcal{B}$ is W-orbit of wts

Each W-orbit of wts contains a unique dominant wt

This dominant wt is unique hw vector in $\mathcal{B}$

(ii) → (i) Let

u = unique hw vector in $\mathcal{B}$

$$x \preceq u \qquad \forall x \in \mathcal{B}$$

Write

$$\lambda = wt(u)$$

$\forall x \in \mathcal{B}$ $wt(x) \preceq \lambda$

with equality iff $x = u$

Recall

$$\Omega = \{ wt(x) \mid x \in \mathcal{B} \}$$

Given $x, y \in B$ st

$$wt(x) = wt(y) \qquad (= \mu)$$

show $x = y$

Assume $x \neq y$. Of all such x, y WLOG

μ is maximal rel $\preceq$

$$\mu \neq \lambda$$

else $x = y = \mu$

x, y not hw

$\exists\, i \in I \quad \exists\, x' \in B$ st

$$\begin{tikzcd} x' \\ x \arrow[u, "f_i"'] \end{tikzcd}$$

$$\langle wt(y), \alpha_i^\vee \rangle = \langle \mu, \alpha_i^\vee \rangle = \langle wt(x), \alpha_i^\vee \rangle = -1$$

So $\exists\, y' \in B$ st

$$\begin{tikzcd} y' \\ y \arrow[u, "f_i"'] \end{tikzcd}$$

obs

$$wt(x') = \mu + \alpha_i = wt(y')$$

$$\mu + \alpha_i \succ \mu$$

$$x' = y' \qquad \text{by maximality of } \mu$$

We have

$$\begin{tikzcd} & x' = y' \arrow[dl, "i"'] \arrow[dr, "i"] & \\ x & & y \end{tikzcd}$$

So

$$x = y$$

Now B is miniscule by prev thm. $\square$

Thm Given a root system Φ and at lattice Λ

Given a crystal $\mathcal{B}$ for Φ, Λ

TFAE:

(i) $\mathcal{B}$ is miniscule;

(ii) $\mathcal{B}$ is feasibly miniscule and

- for $x \in \mathcal{B}$ and distinct $i, j \in I$,

$$\begin{tikzcd} \bullet \arrow[dr, "i"'] & & \bullet \arrow[dl, "j"] \\ & x & \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} & \bullet \arrow[dl, "j"'] \arrow[dr, "i"] & \\ \bullet \arrow[dr, "i"'] & & \bullet \arrow[dl, "j"] \\ & x & \end{tikzcd} \qquad (*)$$

pf (i) $\to$ (ii) Check $*$:

Assume

$$\begin{tikzcd} \bullet \arrow[dr, "i"'] & & \bullet \arrow[dl, "j"] \\ & x & \end{tikzcd}$$

So $\langle x, \alpha_i^\vee \rangle = -1$

$\langle x, \alpha_j^\vee \rangle = -1$

obs

$$\underset{\{1,0,-1\}}{\underset{\rotatebox{90}{$\in$}}{\langle e_i x, \alpha_j^\vee \rangle}} = \langle x + \alpha_i, \alpha_j^\vee \rangle$$

$$= \underset{-1}{\underset{\shortparallel}{\langle x, \alpha_j^\vee \rangle}} + \underbrace{\langle \alpha_i, \alpha_j^\vee \rangle}_{\leq 0}$$

So

$$\langle \alpha_i, \alpha_j^\vee \rangle = 0$$

and

$$\langle e_i x, \alpha_j^\vee \rangle = -1$$

similarly

$$\langle \alpha_j, \alpha_i^\vee \rangle = 0$$

$$\langle e_j x, \alpha_i^\vee \rangle = -1$$

Now

$$\begin{tikzcd}
 & x+\alpha_i+\alpha_j \arrow[dl, "j"'] \arrow[dr, "i"] & \\
x+\alpha_i \arrow[dr, "i"'] & & x+\alpha_j \arrow[dl, "j"] \\
 & x &
\end{tikzcd}$$

(ii) $\to$ (i) Since B is finite

$\exists$ hw vector $u \in B$

By $\maltese$,

$$x \preceq \mu \qquad \forall x \in \mathcal{B}$$

So μ is unique

Now $\mathcal{B}$ is miniscule by prev thm. $\square$

We have been discussing miniscule crystals.

Next we discuss adjoint crystals.

Adjoint crystals

Thm Given root system Φ

weight lattice Λ

Simple roots $\Sigma = \{\alpha_i\}_{i \in I}$

Define a set

$$B = \Phi \cup \tilde{I}$$

where $\tilde{I} = \{ \tilde{i} \mid i \in I \}$

Define a weight function

$$\mathrm{wt}: \quad \begin{array}{ccc} B & \longrightarrow & \Lambda \\ \alpha & \longrightarrow & \alpha \qquad \alpha \in \Phi \\ \tilde{i} & \longrightarrow & 0 \qquad i \in I \end{array}$$

Then B becomes a semi normal crystal for Φ, Λ
"adjoint crystal"

such that for $i \in I$,

$$-\alpha_i \xleftarrow{i} \tilde{h}_i \xleftarrow{i} \alpha_i$$

$$\alpha \xleftarrow{i} \beta \qquad \alpha, \beta \in \Phi \qquad \beta - \alpha = \alpha_i$$

Pf By construction

For $x, y \in \mathcal{B}$ and $i \in I$ st

$$x \xleftarrow{i} y$$

then $\quad wt(y) - wt(x) = \alpha_i$

For $x \in B$ and $i \in I$ show

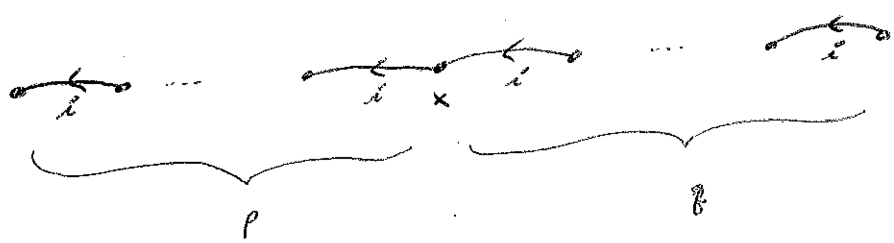
$$\langle wt(x), \alpha_i^\vee \rangle = \varphi_i(x) - \varepsilon_i(x)$$

First assume

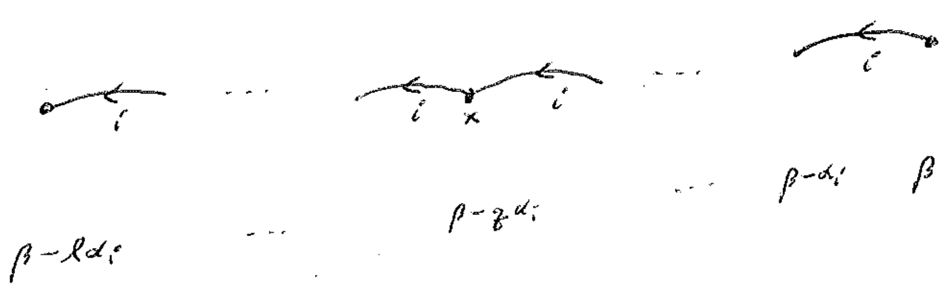
$$x \in \Phi \setminus \{\pm \alpha_i\}$$

Write
$$p = \varphi_i(x)$$
$$q = \varepsilon_i(x)$$

Consider α_i - string thru x :



Let $\ell = p+q$
$=$ length of root string :



obs

$$\beta - l\alpha_i = s_i(\beta)$$
$$= \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$$

So

$$l = \langle \beta, \alpha_i^\vee \rangle$$

$$\langle wt(x), \alpha_i^\vee \rangle \overset{?}{=} \underset{\overset{\|}{p}}{\varphi_i(x)} - \underset{\overset{\|}{q}}{\varepsilon_i(x)}$$

$$wt(x) = X = \beta - q\alpha_i$$

$$\underbrace{\langle \beta - q\alpha_i, \alpha_i^\vee \rangle}_{=\ \langle \beta, \alpha_i^\vee \rangle - q \underbrace{\langle \alpha_i, \alpha_i^\vee \rangle}_{=2}}$$

$$\langle \beta, \alpha_i^\vee \rangle = l = p + q$$

OK

$*$ holds

Next assume

$$x \notin \Phi \setminus \{\pm \alpha_i\}$$

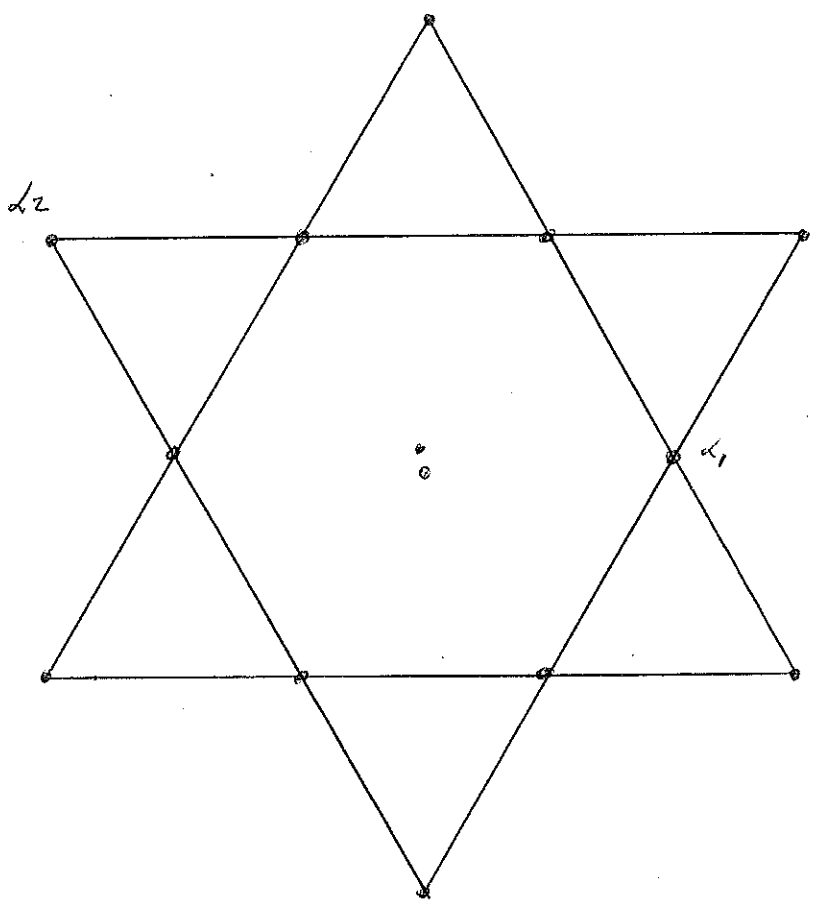
Cases

x	$wt(x)$	$\langle wt(x), \alpha_i^\vee \rangle$	$\varphi_i(x)$	$\varepsilon_i(x)$	$\varphi_i(x) - \varepsilon_i(x)$
α_i	α_i	2	2	0	2
$\tilde{i}$	0	0	1	1	0
$-\alpha_i$	$-\alpha_i$	-2	0	2	-2
$\tilde{j}$ $(j \neq i)$	0	0	0	0	0

In each case $*$ holds.

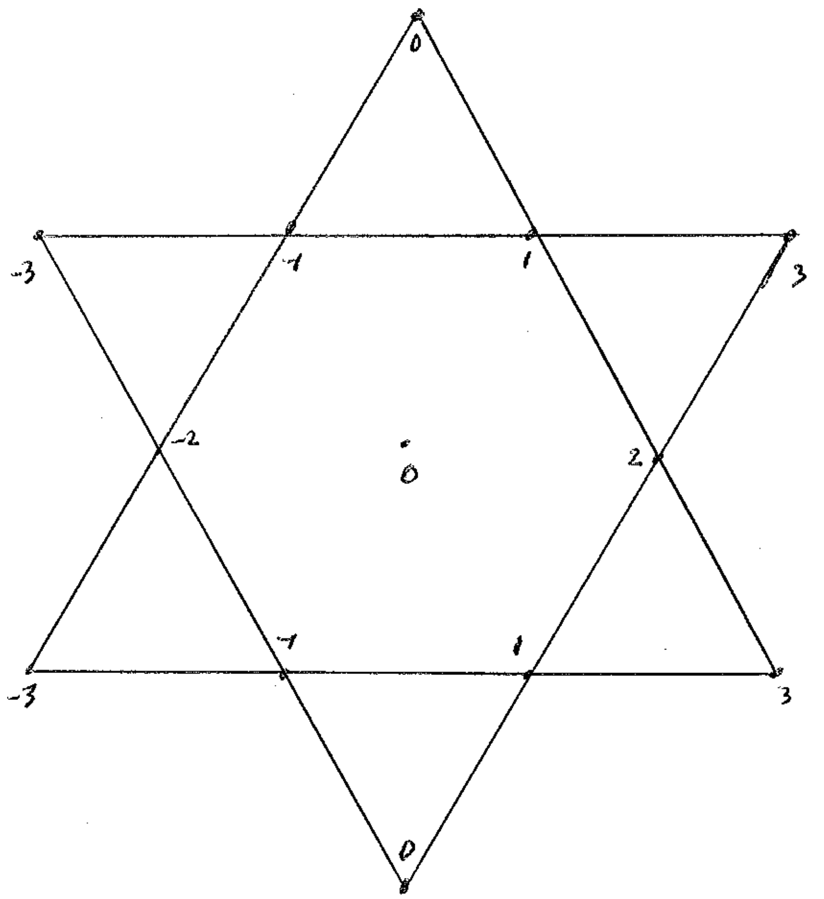
$\square$

Ex $\Phi = G_2$



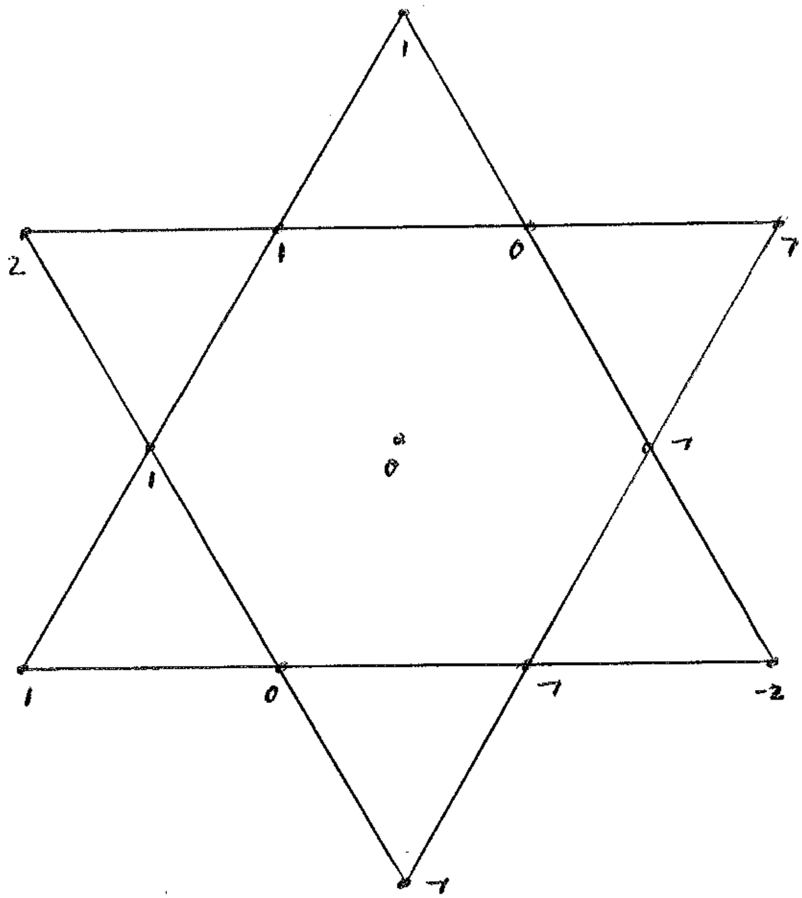
$\Phi = G_2$

showing $\langle \alpha, \alpha_1^\vee \rangle$



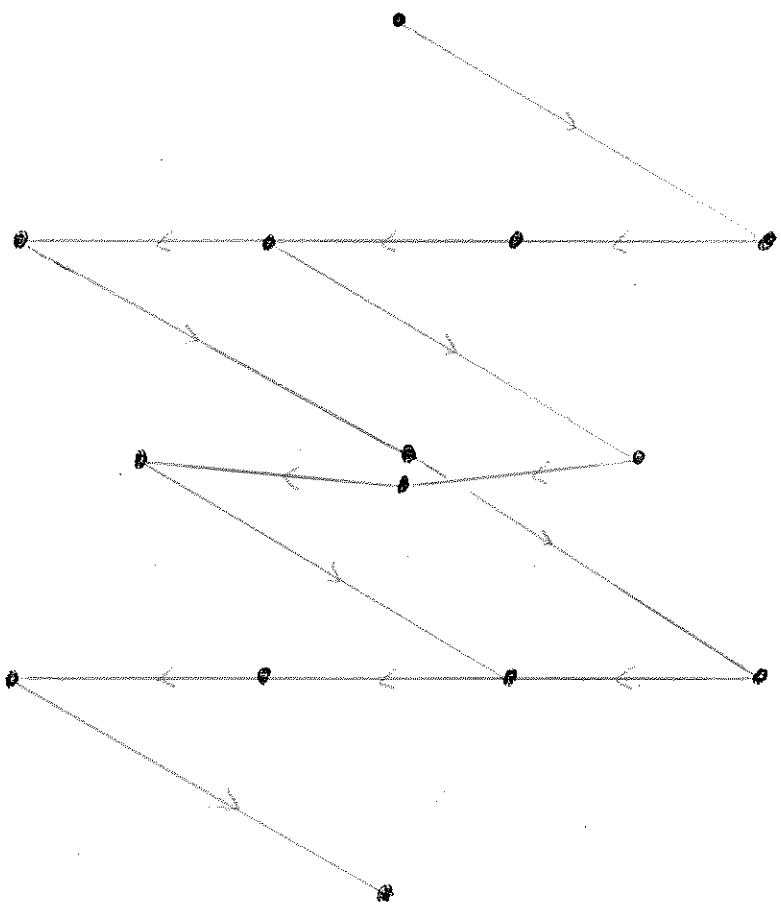
$\Phi = G_2$ 11/1/19

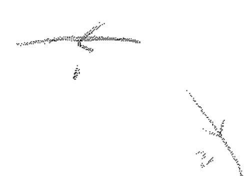
showing $\langle \,\cdot\,, \alpha_2^\vee \rangle$ 15



Ex $\Phi = G_{2,1}$

adjoint crystal



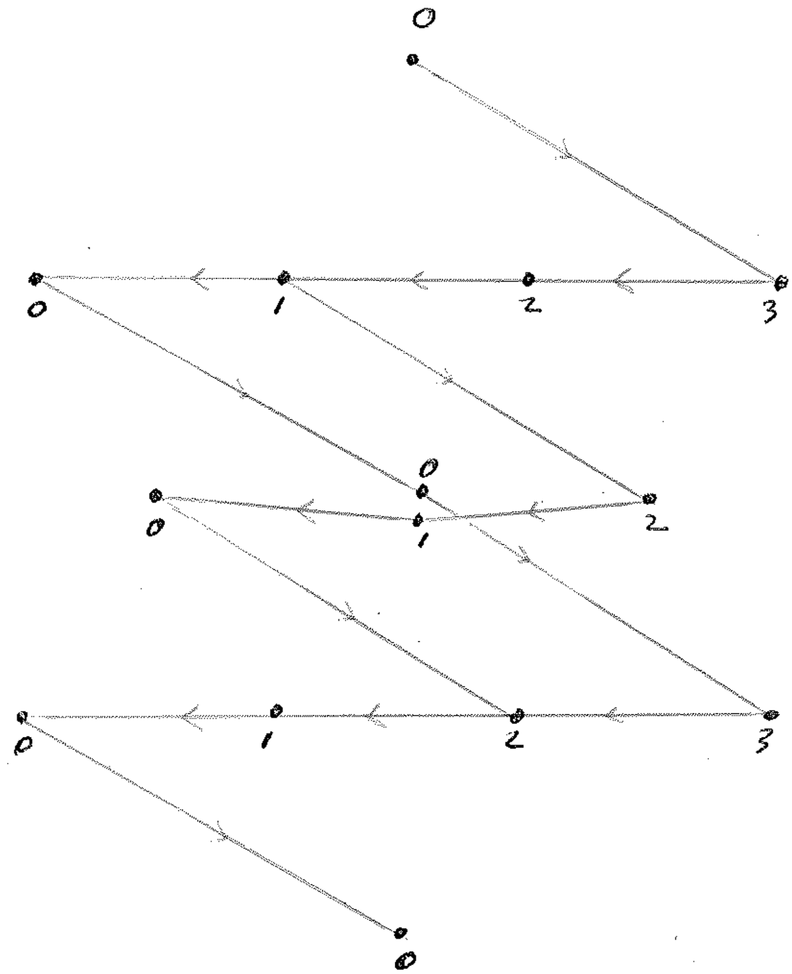


$\Phi = G_2$

adjoint crystal

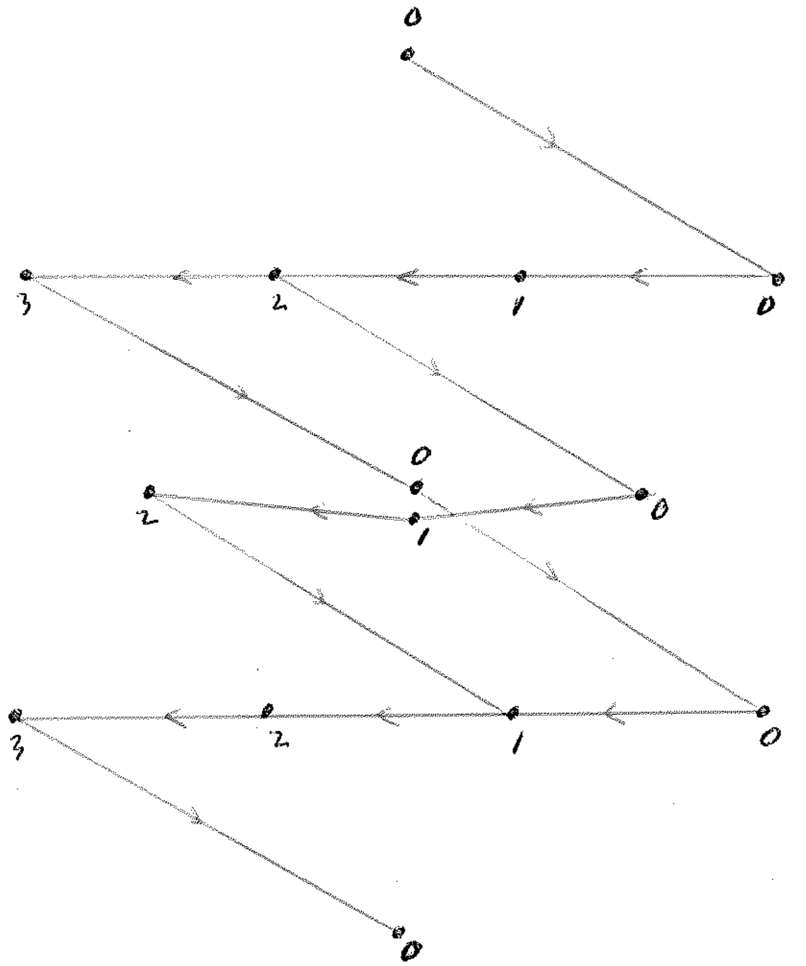
$\Phi = G_2$

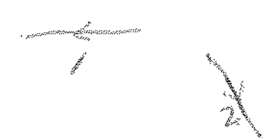
showing φ_1



$\Phi = G_2$

showing $\mathcal{E}_1$





$\Phi = G_2$

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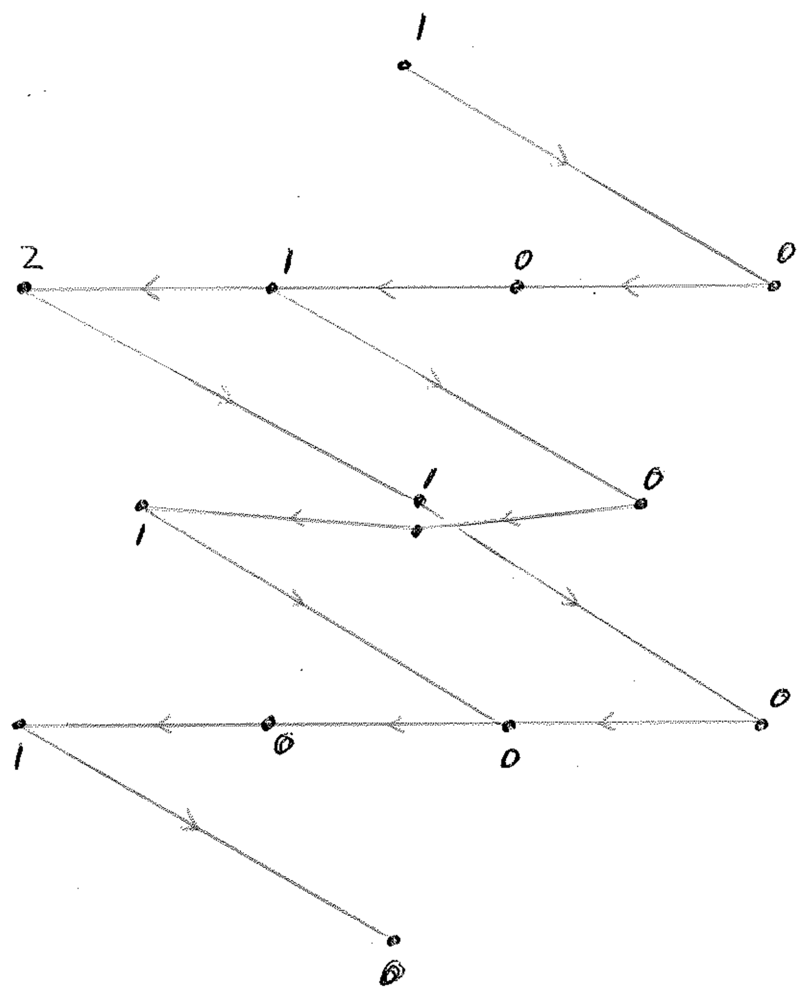
Showing φ_2



$\Phi = G_2$

showing ε_2

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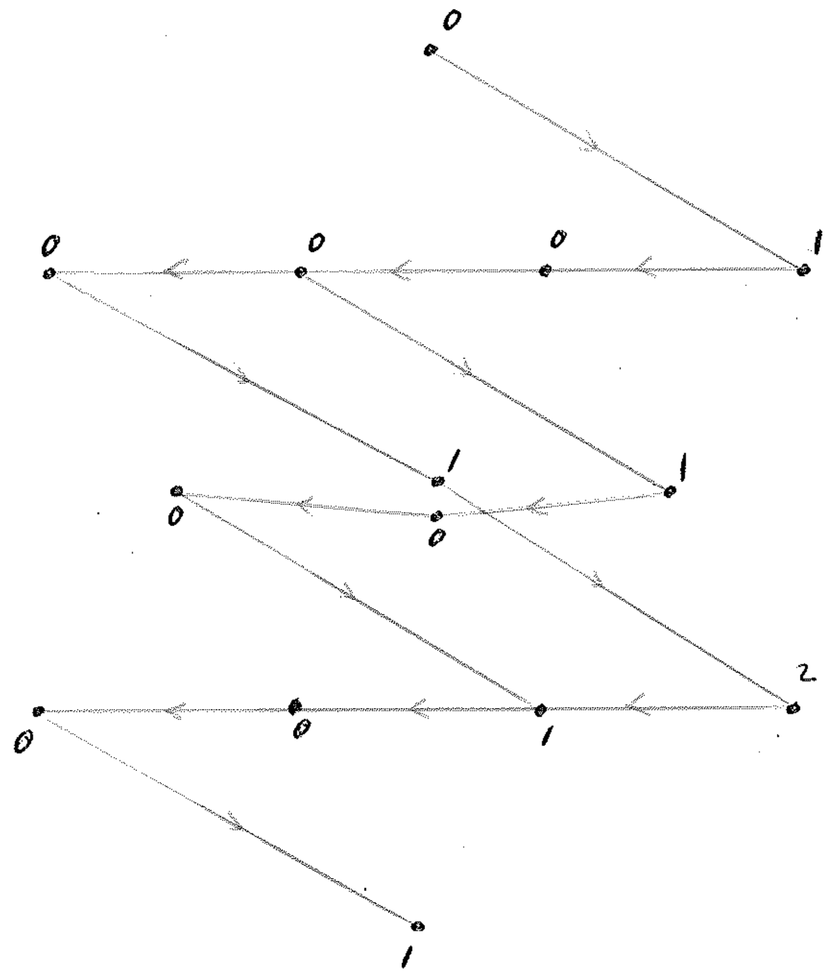
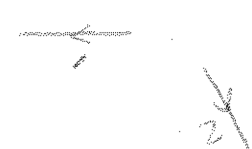




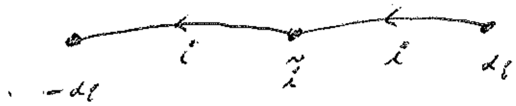
LEM Given adjoint crystal B for a simply-laced root system Φ.
Then B is Stembridge.

pf For $i, j \in I$ $i \neq j$
consider the graph with vertex set B and edges

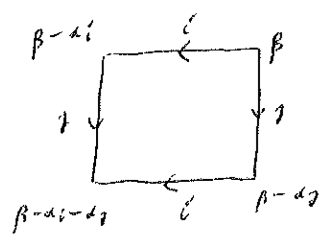
$\xleftarrow{i}$, $\xleftarrow{j}$

Describe connected components of this graph.

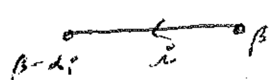
Case $\langle \alpha_i, \alpha_j^\vee \rangle = 0$



$k \in I \setminus \{i, j\}$



$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 1$
$\langle \beta, \alpha_j^\vee \rangle = 1$



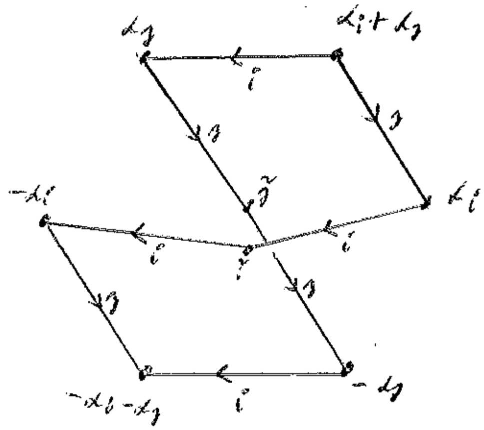
$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 1$
$\langle \beta, \alpha_j^\vee \rangle = 0$

$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 0$
$\langle \beta, \alpha_j^\vee \rangle = 1$

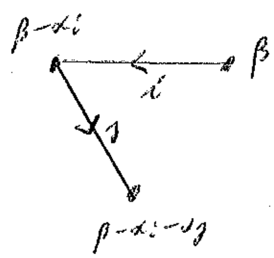
$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 0$
$\langle \beta, \alpha_j^\vee \rangle = 0$

Each connected comp is rectangle w

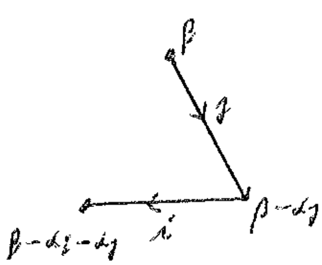
Case $\langle \alpha_i, \alpha_j^\vee \rangle = -1$



$\tilde{k}$ $\qquad$ $k \in I \setminus \{i,j\}$



$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 1$
$\langle \beta, \alpha_j^\vee \rangle = 0$



$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 0$
$\langle \beta, \alpha_j^\vee \rangle = 1$

β

$\beta \in \Phi$
$\langle \beta, \alpha_i^\vee \rangle = 0$
$\langle \beta, \alpha_j^\vee \rangle = 0$

Each connected component is A_2-crystal $\qquad \square$