Lecture 25 Wednesday Oct 30

thm Given simply laced root system Φ
and wt lattice Λ

Assume $\lambda \in \Lambda^+$ is miniscule.

Then the crystal

$$M_\lambda = W\lambda$$

is Stembridge.

pf M_λ is seminormal

For $b \in M_\lambda$

$$\varphi_i(b), \quad \varepsilon_i(b) \in \{0,1\} \qquad i \in I$$

So each root-string has length 0 or 1

Given $i, j \in I$ $i \neq j$

Consider graph with vertex set M_λ and
edges $\xrightarrow{i}$, $\xrightarrow{j}$

let C = connected component of this graph.

Describe C

let x = any hw element in C

First assume $\langle \alpha_i, \alpha_j \rangle = 0$

the possibilites for C are

C :

	$\varphi_j(x) = 0$	$\varphi_j(x) = 1$
$\varphi_i(x) = 0$	x	$\begin{array}{c} x \\ \downarrow j \\ x-\alpha_j \end{array}$
$\varphi_i(x) = 1$	$x-\alpha_i \xleftarrow{i} x$	$\begin{array}{ccc} x-\alpha_i & \xleftarrow{i} & x \\ \downarrow j & & \downarrow j \\ x-\alpha_i-\alpha_j & \xleftarrow{i} & x-\alpha_j \end{array}$

In each case, C is rectangle.

Next assume

$$\langle \alpha_i, \alpha_j \rangle = -1$$

The possibilities for C are

C:

	$\varphi_j(x) = 0$	$\varphi_j(x) = 1$
$\varphi_i(x) = 0$	x	$x \xrightarrow{j} x-\alpha_j \xrightarrow{i} x-\alpha_i-\alpha_j$
$\varphi_i(x) = 1$	$x \xrightarrow{i} x-\alpha_i \xrightarrow{j} x-\alpha_i-\alpha_j$	Imposs since $\alpha_i+\alpha_j \in \Phi$, $\langle x, \alpha_i+\alpha_j \rangle = 2 \notin \{1, 0, -1\}$

In each case C is A_2-crystal

Result follows. $\square$

Earlier we constructed miniscule crystals

$$B_{\bar{\omega}_r} \quad \text{for } B_r$$

$$B_{\bar{\omega}_r},\ B_{\bar{\omega}_m} \quad \text{for } D_{r+1}$$

Next we construct a virtual crystal

$$B \leq \hat{B}$$

where

$$B = B_{\bar{\omega}_r} \quad \text{for } B_r$$

$$\hat{B} = \underset{B_1}{\underset{\parallel}{B_{\bar{\omega}_r}}} \otimes \underset{B_2}{\underset{\parallel}{B_{\bar{\omega}_m}}} \quad \text{for } D_{r+1}$$

We just showed

B_1, B_2 are Stembridge

So $\hat{B} = B_1 \otimes B_2$ is Stembridge

For $1 \le i \le m$ and

$$x \otimes y \in B_1 \otimes B_2$$

find

$$\hat{f}_i(x \otimes y) \qquad *$$

We have $\varphi_i(x), \quad \varphi_i(y) \in \{0, 1\}$

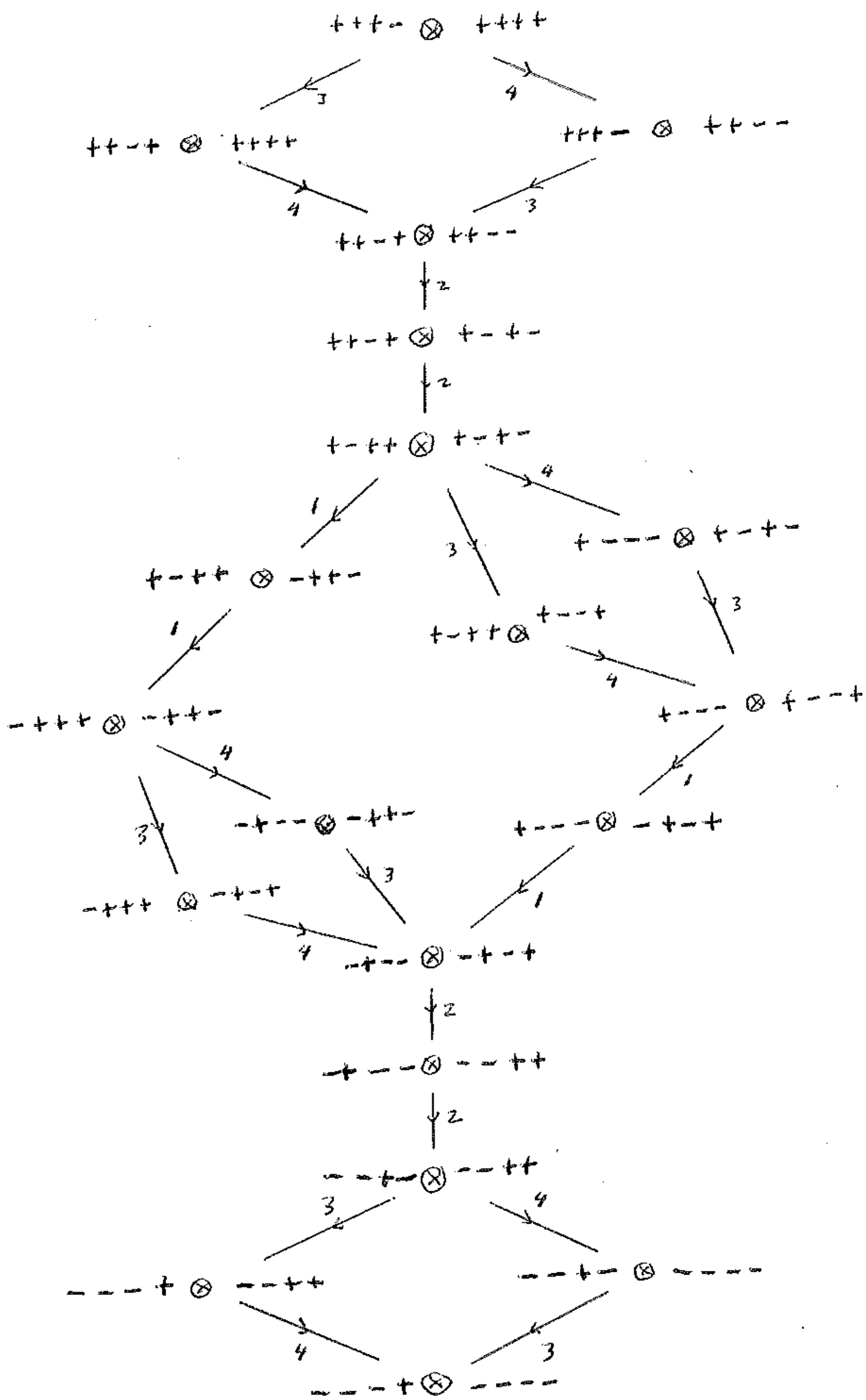
Using signature rule, possibilities for $*$ are

$\hat{f}_i(x \otimes y)$:

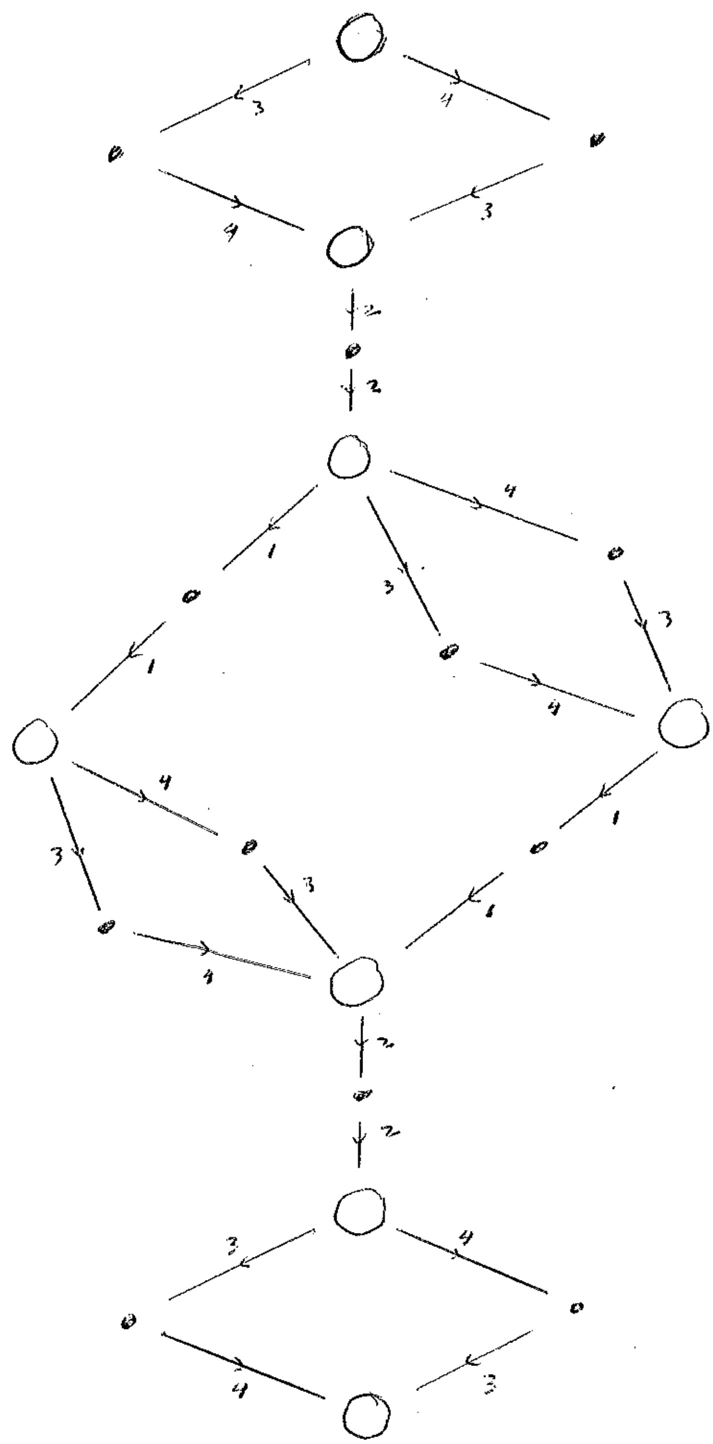
	$\varphi_i(y) = 0$	$\varphi_i(y) = 1$
$\varphi_i(x) = 0$	ϕ	$x \otimes f_i(y)$
$\varphi_i(x) = 1$	$f_i(x) \otimes y$	$x \otimes f_i(y)$

ex $r=3$

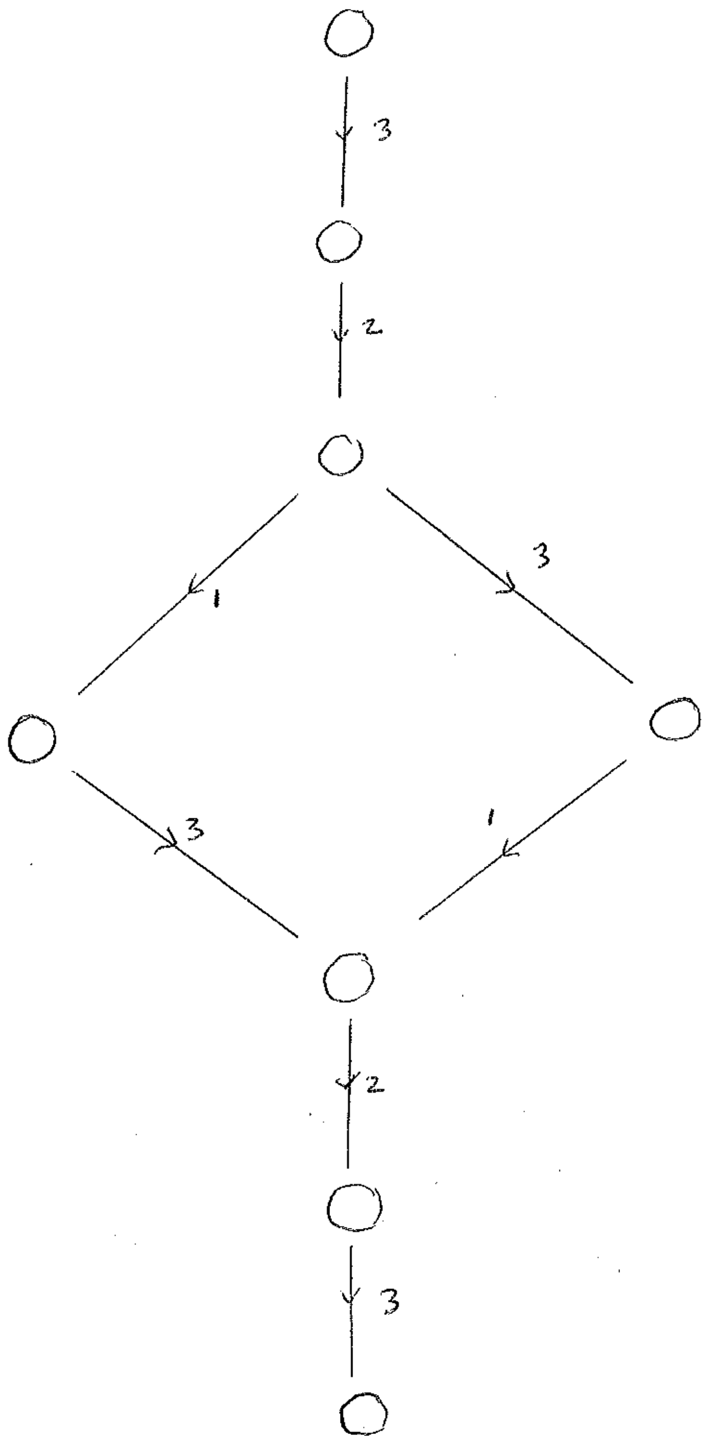
In $\overset{n}{B}$



$r=3$ in $\hat{B}$ aligned nodes denoted ○



$r=3$ in $\hat{B}$ showing virtual operators acting on aligned nodes



This is crystal graph for B ✓

Next goal:
characterize the miniscule crystals

Given root system Φ. wt lattice Λ

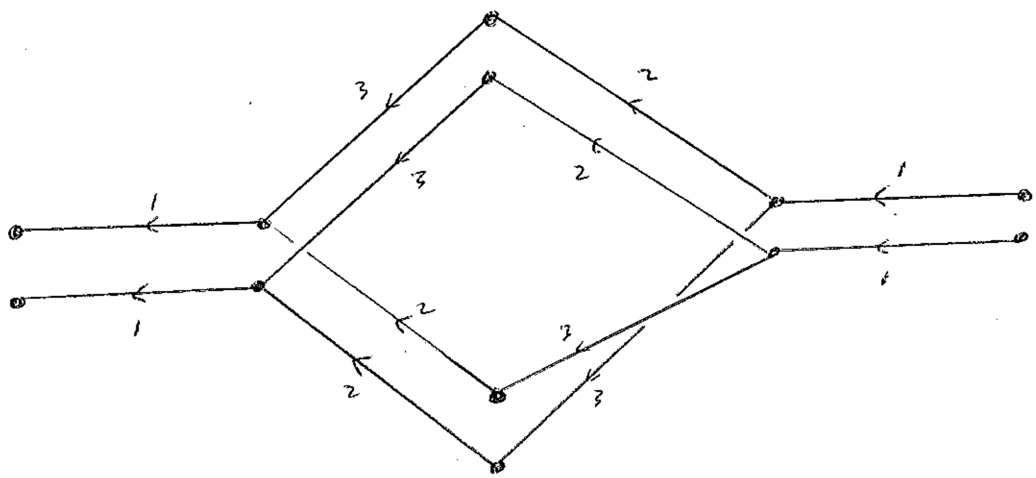
Given crystal B for Φ, Λ

Call B <u>feasibly miniscule</u> whenever

- B is finite, connected, seminormal
- For $b \in B$ and $i \in I$,

  $\varphi_i(b), \ \varepsilon_i(b) \in \{0, 1\}$

· Example of a feasibly miniscule D_3-crystal that is not miniscule



e_3

wt: $-e_1$ $-e_2$ e_2 e_1

$-e_3$

Given least minusc crystal B for Φ, Λ.

Each root string has length $0, 1$.

For $i \in I$,

	length 1		length 0
	$\bullet \xleftarrow{\;i\;}$	$\bullet$	$\bullet$
φ_i	0	1	0
ε_i	1	0	0

Recall for $b \in B$,

$$\langle wt(b), \alpha_i^\vee \rangle = \varphi_i(b) - \varepsilon_i(b)$$

α_i-root string cont b	$\langle wt(b), \alpha_i^\vee \rangle$
$\bullet \xleftarrow{\;i\;} \bullet\, b$	1
$b\, \bullet \xleftarrow{\;i\;} \bullet$	-1
$\bullet\, b$	0

So for $b \in B$,

$$\langle wt(b), \alpha_i^\vee \rangle \in \{0, 1, -1\}$$

Define

$$\Omega = \{ wt(b) \mid b \in B \}$$

We saw earlier that

Ω is invar under Weyl gp W of Φ

So Ω = union of W-orbits

LEM Ω is a single W-orbit

pf $\forall\, x, y \in B$ and $i \in I$ st

$$x \xleftarrow{\;i\;} y$$

then $\langle wt(y), \alpha_i^\vee \rangle = 1$

So
$$\begin{aligned} wt(x) &= wt(y) - \alpha_i \\ &= wt(y) - \langle wt(y), \alpha_i^\vee \rangle \alpha_i \\ &= s_i(wt(y)) \end{aligned}$$

$wt(x)$, $wt(y)$ in same W-orbit of Ω

B is connected so Ω is single W-orbit $\square$

LEM With above notation,

the elements in Ω are miniscule

pf For $\lambda \in \Omega$ and $\alpha \in \Phi$ show

$$\langle \lambda, \alpha^\vee \rangle \in \{1, 0, -1\}$$

$\exists$ simple root α_i and $\exists\, w \in W$ st

$$\alpha = w(\alpha_i)$$

So

$$\alpha^\vee = w(\alpha_i^\vee)$$

Now

$$\begin{aligned}\langle \lambda, \alpha^\vee \rangle &= \langle \lambda, w(\alpha_i^\vee) \rangle \\ &= \langle w^{-1}(\lambda), \alpha_i^\vee \rangle \qquad w^{-1}(\lambda) \in \Omega \\ &\in \{1, 0, -1\} \qquad \text{by } ☆\end{aligned}$$

We have shown,

Ω is a W-orbit of miniscule weights.

So Ω becomes a miniscule crystal for Φ, Λ

LEM The map

$$\mathrm{wt}: \begin{array}{ccc} \mathcal{B} & \longrightarrow & \Omega \\ b & \longrightarrow & \mathrm{wt}(b) \end{array}$$

is a strict crystal morphism.

pf Compare the def of miniscule crystal with the def of strict crystal morphism. $\square$