Lecture 24 Monday Oct 28

Ex $\Phi = D_r \qquad \lambda = \varpi_r = \frac{e_1 + \dots + e_r}{2}$

$$M_\lambda = \left\{ \frac{1}{2} \sum_{i=1}^{r} \varepsilon_i e_i \;\middle|\; \varepsilon_i \in \{+,-\}, \quad |\{ i \mid \varepsilon_i = - \}| \text{ even} \right\}$$

$\xleftarrow[i]{}$ as before

$\Phi = D_r \qquad \lambda = \varpi_r \qquad M_\lambda . \qquad r = 4$

$$
\begin{tikzcd}
 & {+}{+}{+}{+} \arrow[d, "4"] & \\
 & {+}{+}{-}{-} \arrow[d, "2"] & \\
 & {+}{-}{+}{-} \arrow[dl, "1"'] \arrow[dr, "3"] & \\
{-}{+}{+}{-} \arrow[dr, "3"'] & & {+}{-}{-}{+} \arrow[dl, "1"] \\
 & {-}{+}{-}{+} \arrow[d, "2"] & \\
 & {-}{-}{+}{+} \arrow[d, "4"] & \\
 & {-}{-}{-}{-} &
\end{tikzcd}
$$

$\Phi = D_r$ $\qquad \lambda = \overline{\omega}_r$ $\qquad M_\lambda$ $\qquad r = 5$

$$
\begin{tikzcd}[column sep=small]
 & & {+++++} \arrow[d, "5"] & & \\
 & & {+++--} \arrow[d, "3"] & & \\
 & & {++-+-} \arrow[dl, "2"'] \arrow[dr, "4"] & & \\
 & {+-++-} \arrow[dl, "1"'] \arrow[dr, "4"] & & {++--+} \arrow[dl, "2"] & \\
{-+++-} \arrow[dr, "4"] & & {+-+-+} \arrow[dl, "1"] \arrow[dr, "3"] & & \\
 & {-++-+} \arrow[dr, "3"'] & & {+--++} \arrow[dl, "1"] \arrow[dr, "5"] & \\
 & & {-+-++} \arrow[dl, "2"'] \arrow[dr, "5"] & & {+----} \arrow[dl, "1"] \\
 & {--+++} \arrow[dr, "5"] & & {-+---} \arrow[dl, "2"] & \\
 & & {--+--} \arrow[d, "3"] & & \\
 & & {---+-} \arrow[d, "4"] & & \\
 & & {----+} & &
\end{tikzcd}
$$

Compare

- M_λ for D_r $\quad \lambda = \overline{\omega_{r-1}}$
- M_λ for D_r $\quad \lambda = \overline{\omega_r}$

To go from one to the other, need

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_r \longrightarrow \varepsilon_1 \varepsilon_2 \cdots \overline{\varepsilon_r}$$

and swap edge labels

$$\xleftarrow[r-1]{} \iff \xleftarrow[r]{}$$

Describe E_6, E_7, E_8

Start with E_8

Guess $D_8 \subseteq E_8$

$$D_8 = \{ \pm e_i \pm e_j \mid 1 \le i, j \le 8 \quad i \ne j \}$$

$$E_8 = D_8 \cup \Delta$$

Find Δ

$\exists \ \alpha \in \Delta$

$$\langle \alpha, \beta \rangle \in \{1, 0, -1\} \quad \forall \beta \in D_8$$
$$\in \mathbb{Z}$$

So

$$\alpha \in \Lambda \ (= \Lambda_{ac} \text{ for } D_8)$$
$$= \left\{ \sum_{i=1}^{8} a_i e_i \mid a_i \in \mathbb{Z} \right\} \cup \left\{ \tfrac{1}{2} \sum_{i=1}^{8} a_i e_i \mid a_i \in \mathbb{Z}_{odd} \right\}.$$

$*$

Also $\langle \alpha, \alpha \rangle = 2$

First assume

$$\alpha = \sum_{i=1}^{8} a_i e_i \qquad a_i \in \mathbb{Z}$$

$*$ gives

$$2 = \sum_{i=1}^{8} a_i^2$$

$\alpha \in D_8$ nothing new

Next assume

$$\alpha = \frac{1}{2}\sum_{i=1}^{8} a_i e_i \qquad a_i \text{ odd}$$

$*$ gives

$$8 = \sum_{i=1}^{8} a_i^2$$

$$a_i = \mp 1 \qquad 1 \le i \le 8$$

Define

$$\Delta^+ = \left\{ \frac{1}{2}\sum_{i=1}^{8} a_i e_i \;\middle|\; a_i = \mp 1, \quad \prod_{i=1}^{8} a_i = 1 \right\}$$

$$\Delta^- = \left\{ \frac{1}{2}\sum_{i=1}^{8} a_i e_i \;\middle|\; a_i = \mp 1, \quad \prod_{i=1}^{8} a_i = -1 \right\}$$

We have
$\Delta \subseteq \Delta^+ \cup \Delta^-$

Since $D_8 \subseteq E_8$

Weylgroup $W(D_8) \subseteq W(E_8)$

Recall
$W(D_8)$: perms of $e_1, e_2, \ldots e_8$
and even sign changes

Each of Δ^+, Δ^- is an orbit for $W(D_8)$

So $\Delta = \Delta^+$

or $\Delta = \Delta^-$

or $\Delta = \Delta^+ \cup \Delta^-$

Replacing $e_8 \to -e_8$ (diagram aut for D_8)

if nec. WLOG

$$\Delta^+ \subseteq \Delta$$

So $$\frac{1}{2}\sum_{i=1}^{8} e_i \in \Delta$$

δ ''

For $\beta \in D_8$ check

$$\langle \beta, \delta \rangle \in \{1, 0, -1\}$$

write $\beta = \pm e_i \pm e_j$

$$\langle \beta, \delta \rangle = \pm \tfrac{1}{2} \pm \tfrac{1}{2}$$
$$\in \{1, 0, -1\} \quad \checkmark$$

For $\beta \in \Delta^+ \cup \Delta^-$

find $\langle \beta, d \rangle$

write $\beta = \frac{1}{2} \sum_{i=1}^{8} a_i e_i$

define

$$p = \left| \{ i \mid 1 \le i \le 8, \; a_i = 1 \} \right|$$

$$n = \left| \{ i \mid 1 \le i \le 8, \; a_i = -1 \} \right|$$

So

$$p + n = 8$$

$$\frac{p - n}{4} = \langle \beta, d \rangle$$

So

$$\frac{p + n}{4} = 2$$

$$\frac{p - n}{4} = \langle \beta, d \rangle$$

$$\langle \beta, d \rangle = 2 - \frac{n}{2}$$

For even $n \in \{0, 2, 4, 6, 8\}$,

$$\langle \beta, d \rangle \in \{ \pm 2, \pm 1, 0 \}$$

For odd $n \in \{1, 3, 5, 7\}$

$$\langle \beta, d \rangle \in \{ \pm 3/2, \pm 1/2 \} \quad \notin \mathbb{Z}$$

We conclude

$$\Delta = \Delta^+$$

If our guess $D_8 \subseteq E_8$ is correct, then

$$E_8 = D_8 \cup \Delta^+$$

LEM

$$E_8 = D_8 \cup \Delta^+$$
$$\shortparallel$$
$$\Phi$$

pf check Φ is root system:

We saw

$$\langle \alpha, \alpha \rangle = 2 \qquad \alpha \in \Phi$$
$$\langle \alpha, \beta \rangle \in \{\pm 2, \pm 1, 0\} \qquad \alpha, \beta \in \Phi$$

Also for $v = \sum_{i=1}^{8} a_i e_i \qquad a_i \in \mathbb{R}$

if $\langle v, v \rangle = 2$ and $\langle v, \alpha \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi$

then $v \in \Phi$

Now for $\alpha, \beta \in \Phi$ reflection

$$r_\alpha(\beta) \in \Phi$$

Since

$$\langle r_\alpha(\beta), r_\alpha(\beta)\rangle = \langle \beta, \beta\rangle = 2$$

and $\forall\, \gamma \in \Phi$

$$\langle r_\alpha(\beta), \gamma\rangle = \langle \beta - \langle \alpha,\beta\rangle\alpha, \gamma\rangle$$

$$= \underbrace{\langle \beta,\gamma\rangle}_{\in \mathbb{Z}} - \underbrace{\langle \alpha,\beta\rangle}_{\in \mathbb{Z}}\underbrace{\langle \alpha,\gamma\rangle}_{\in \mathbb{Z}}$$

$$\in \mathbb{Z}$$

So Φ is root system

By constr Φ is simply-laced, rank 8

Φ is indec since it contains D_8

$\Phi \neq A_8$ (wrong size)

So $\Phi = E_8$ $\square$

Note One checks

simply connected wt lattice for E_8

$=$ root lattice for E_8

$$= \left\{ \sum_{i=1}^{8} a_i e_i \;\middle|\; a_i \in \mathbb{Z},\ a_1 + \cdots + a_8 \text{ even} \right\}$$

$$\cup \left\{ \frac{1}{2} \sum_{i=1}^{8} a_i e_i \;\middle|\; a_i \in \mathbb{Z}_{odd},\ 4 \mid a_1 + \cdots + a_8 \right\}$$

LEM E_8 has no miniscule wt.

pf Assume $\exists$ minisc wt λ for E_8

So $\langle \lambda, r \rangle \in \{1, 0, -1\}$ $\forall r \in E_8$ $*$

λ is minisc wt for D_8

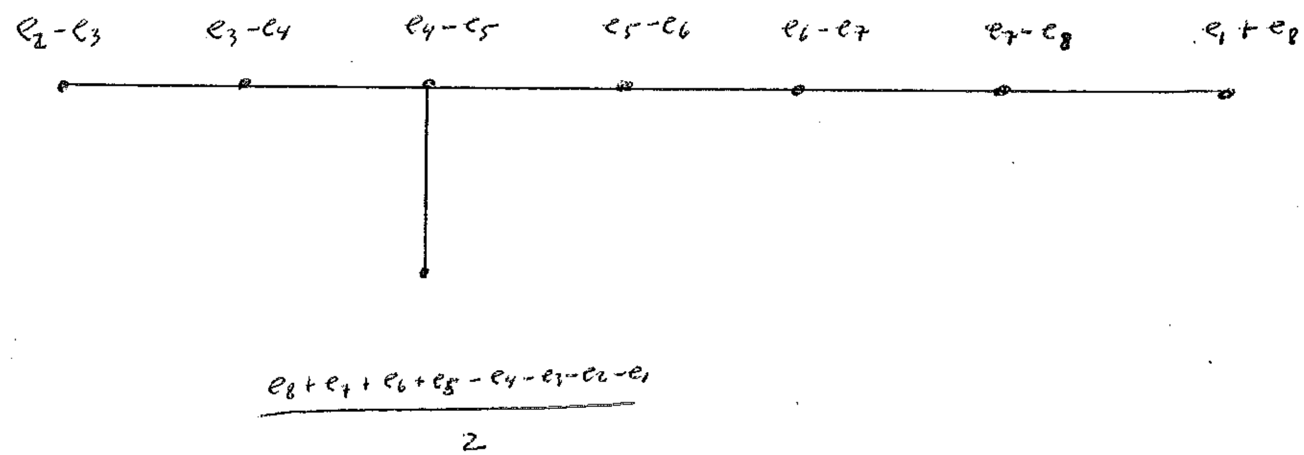
Cases (up to $W(D_8)$):

λ	$\langle \lambda, d \rangle$
e_1	$\frac{1}{2}$
$\frac{e_1 + \dots + e_7 - e_8}{2}$	$3/2$
$\frac{e_1 + \dots + e_8}{2}$	2

$*$ fails in each case

$\square$

Simple roots for E_8



Describe E_7

let vector space

U = span of the simple roots of E_8 besides $e_1 + e_8$

= orthog complement of $d = \frac{e_1 + \cdots + e_8}{2}$

$E_7 = E_8 \cap U$

$= \{ x \in E_8 \mid \langle x, d \rangle = 0 \}$

$= \{ \underbrace{e_i - e_j \mid 1 \le i, j \le 8,\ i \ne j}_{A_7} \} \cup \{ \frac{1}{2} \sum_{i=1}^{8} a_i e_i \mid a_i = \mp 1, \text{ four } 1, \text{ four } -1 \}$

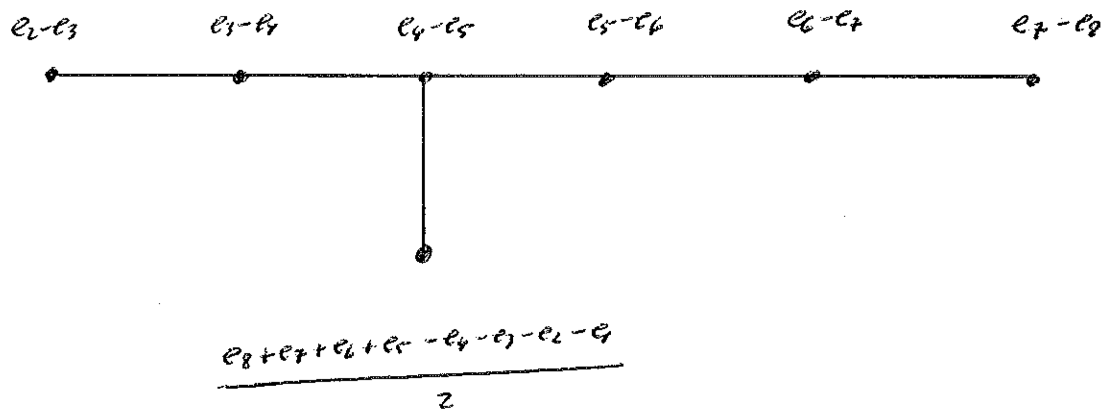
the Weyl group of E_8 is trans on E_8

Also $d \in E_8$

So $\forall \alpha \in E$.

$E_7 = \{ x \in E_8 \mid \langle x, \alpha \rangle = 0 \}$

Simple roots for E_7



Minisc wts of E_7 are

$$e_i + e_j - \frac{e_1 + \cdots + e_8}{4} \qquad 1 \le i, j \le 8 \quad i \ne j$$

and their opposites

Describe E_6

Let vectr space

$\underline{U}$ = span of simple roots in E_8 besides $e_1 + e_8$, $e_7 - e_8$

= orthog complement of e_1+e_8, d

$$E_6 = E_8 \cap \underline{U}$$

$$= \left\{ x \in E_8 \mid \langle x, e_1+e_8 \rangle = 0 \text{ and } \langle x, d \rangle = 0 \right\}$$

$$= \underbrace{\left\{ e_i - e_j \mid 2 \le i,j \le 7,\ i \ne j \right\}}_{A_5} \cup \left\{ \tfrac{1}{2}\sum_{i=1}^{8} a_i e_i \mid a_i = \mp 1,\ \text{four } 1,\ \text{four } -1,\ a_1, a_8 \text{ opp sign} \right\}$$

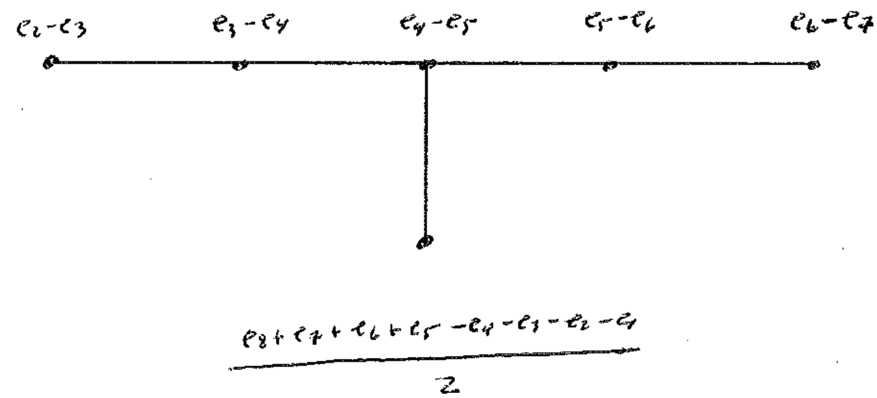
In fact for any $\alpha, \beta \in E_8$ st $\langle \alpha, \beta \rangle = \pm 1$,

$$E_6 = \left\{ x \in E_8 \mid \langle x, \alpha \rangle = 0 \text{ and } \langle x, \beta \rangle = 0 \right\}$$

Ex Find miniscule wts for E_6

10/28/19

Simple roots for E_6



Describe F_4

$$\Phi = B_4 \cup \left\{ \frac{1}{2} \sum_{i=1}^{4} a_i e_i \,/\, a_i = \mp 1 \right\}$$

Simple roots of F_4: