LEM

(i) For $\Phi = B_r$ and $k \geq 1$,

$\mathbb{B}^{\otimes k}$ does not contain an element with wt $\overline{\omega_r}$

(ii) For $\Phi = D_r$ and $k \geq 1$,

$\mathbb{B}^{\otimes k}$ does not contain an element with wt $\overline{\omega_{r-1}}$

or $\overline{\omega_r}$

pf (i) For each element of $\mathbb{B}$ its weight is contained in

$$\sum_{i=1}^{r} \mathbb{Z} e_i$$

So for each element of $\mathbb{B}^{\otimes k}$ its weight is contained in

$$\sum_{i=1}^{r} \mathbb{Z} e_i$$

But $\overline{\omega_r} = \frac{e_1 + \cdots + e_r}{2}$

(ii) Similar since

$$\overline{\omega_{r-1}} = \frac{e_1 + \cdots + e_{r-1} - e_r}{2}, \qquad \overline{\omega_r} = \frac{e_1 + \cdots + e_r}{2}$$

$\square$

To handle the above "missing cases" we consider the "miniscule weights"

Given root system Φ and wt lattice Λ.

For $\lambda \in \Lambda$,

Call λ <u>miniscule</u> whenever

$$\langle \lambda, \alpha^\vee \rangle \in \{1, 0, -1\} \qquad \forall \alpha \in \Phi$$

Assume λ is minisc. Then

$$w(\lambda) \text{ is minisc} \qquad \forall w \in W$$

To find the minisc wts, suffice to find the minisc dominant wts. Turns out

Φ	minisc dom wts
A_r	$\varpi_k \quad 1 \le k \le r$
B_r	ϖ_r
C_r	ϖ_1
D_r	$\varpi_1, \ \varpi_{r-1}, \ \varpi_r$
E_6	$\varpi_1, \ \varpi_6$
E_7	ϖ_7
E_8	none
F_4	none
G_2	none

LEM Assume $\lambda \in \Lambda^+$ is miniscule.

Then the W-orbit

$$B \doteq W\lambda$$

becomes a crystal for Φ, Λ st for $\mu \in B$,

$$wt(\mu) = \mu$$

and for $i \in I$,

Case	$\varphi_i(\mu)$	$\varepsilon_i(\mu)$	$f_i(\mu)$	$e_i(\mu)$
$\langle \mu, \alpha_i^\vee \rangle = 1$	1	0	$s_i(\mu) = \mu - \alpha_i$	ϕ
$\langle \mu, \alpha_i^\vee \rangle = 0$	0	0	ϕ	ϕ
$\langle \mu, \alpha_i^\vee \rangle = -1$	0	1	ϕ	$s_i(\mu) = \mu + \alpha_i$

The Crystal B is connected and semi normal, with unique hw vector λ. Call B the <u>Miniscule crystal M_λ</u>

pf Routine check. $\square$

Note For $\Phi = A_r$ type $GL(r+1)$

and $\lambda = \overline{\omega_k}$ then

M_λ iso $B_{(1^k)}$

obs $(1^k) = e_1 + e_2 + \cdots + e_k = \overline{\omega_k}$

So M_λ iso $B_{\overline{\omega_k}}$

For $\Phi = C_r, D_r$ and $\lambda = \overline{\omega_1}$ then

M_λ iso standard crystal $B_{\overline{\omega_1}}$

Ex For $\Phi = B_r$ and

$$\lambda = \overline{w_r} = \frac{e_1 + \cdots + e_r}{2}$$

$$M_\lambda = \left\{ \underbrace{\frac{1}{2} \sum_{i=1}^{r} \varepsilon_i e_i}_{\text{abbrev}} \;\middle|\; \varepsilon_i \in \{+,-\} \quad 1 \le i \le r \right\}$$

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_r$$

For $1 \le i \le r-1$

$$** \cdots * \underset{i}{-} \underset{i+1}{+} * \cdots ** \xleftarrow{\;i\;} ** \cdots * \underset{i}{+} \underset{i+1}{-} * \cdots **$$

$* =$ arb

$$** \cdots ** - \xleftarrow{\;r\;} ** \cdots ** +$$

$\Phi = B_r \qquad \lambda = \overline{\omega_r} \qquad M_\lambda \qquad r = 3$

$$
\begin{tikzcd}
 & {+}{+}{+} \arrow[d, "3"] & \\
 & {+}{+}{-} \arrow[d, "2"] & \\
 & {+}{-}{+} \arrow[dl, "1"'] \arrow[dr, "3"] & \\
{-}{+}{+} \arrow[dr, "3"] & & {+}{-}{-} \arrow[dl, "1"'] \\
 & {-}{+}{-} \arrow[d, "2"] & \\
 & {-}{-}{+} \arrow[d, "3"] & \\
 & {-}{-}{-} &
\end{tikzcd}
$$

$\Phi = B_r \qquad \lambda = \varpi_r \qquad M_\lambda \qquad$ for $r=4$

$$
\begin{tikzcd}
 & & ++++ \arrow[d, "4"] & & \\
 & & +++- \arrow[d, "3"] & & \\
 & & ++-+ \arrow[dl, "2"'] \arrow[dr, "4"] & & \\
 & +-++ \arrow[dl, "1"'] \arrow[dr, "4"] & & ++-- \arrow[dl, "2"] & \\
-+++ \arrow[dr, "4"'] & & +-+- \arrow[dl, "1"'] \arrow[dr, "3"] & & \\
 & -++- \arrow[dr, "3"'] & & +--+ \arrow[dl, "1"'] \arrow[dr, "4"] & \\
 & & -+-+ \arrow[dl, "2"'] \arrow[dr, "4"] & & +--- \arrow[dl, "1"] \\
 & --++ \arrow[dr, "4"'] & & -+-- \arrow[dl, "2"] & \\
 & & --+- \arrow[d, "3"] & & \\
 & & ---+ \arrow[d, "4"] & & \\
 & & ---- & &
\end{tikzcd}
$$

$\Phi = B_r$, cont

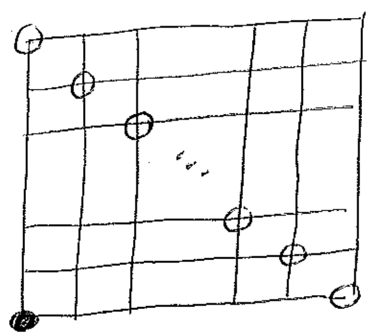
We just drew some posets.

We now interpret these posets for general r

vertices in M_λ $\quad\lambda = \varpi_r$

are in bijection with the lattice paths of length r

from ● to ○ below:



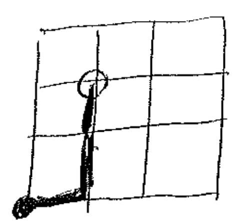
$r \times r$ square

$+$ means go north

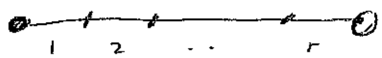
$-$ means go east

So for $r=3$

$-++$ gives



For an r-path Label edges:



For $x, y \in M_\lambda$

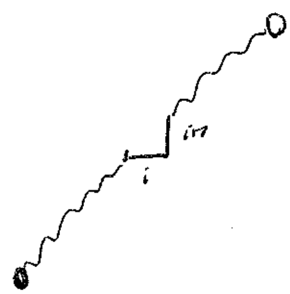
$$x \xleftarrow{\ i\ } y$$

whenever

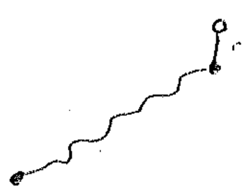
y:

$1 \le i \le r-1$

x:

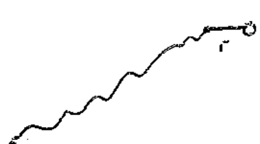


y:



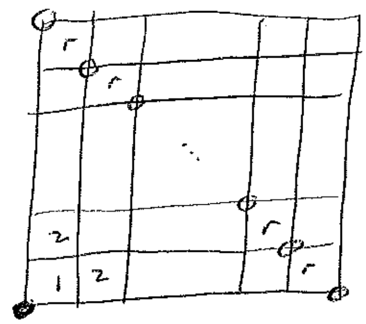
$i = r$

x:



Label the Lattice blocks $1, 2, \ldots r$

as follows



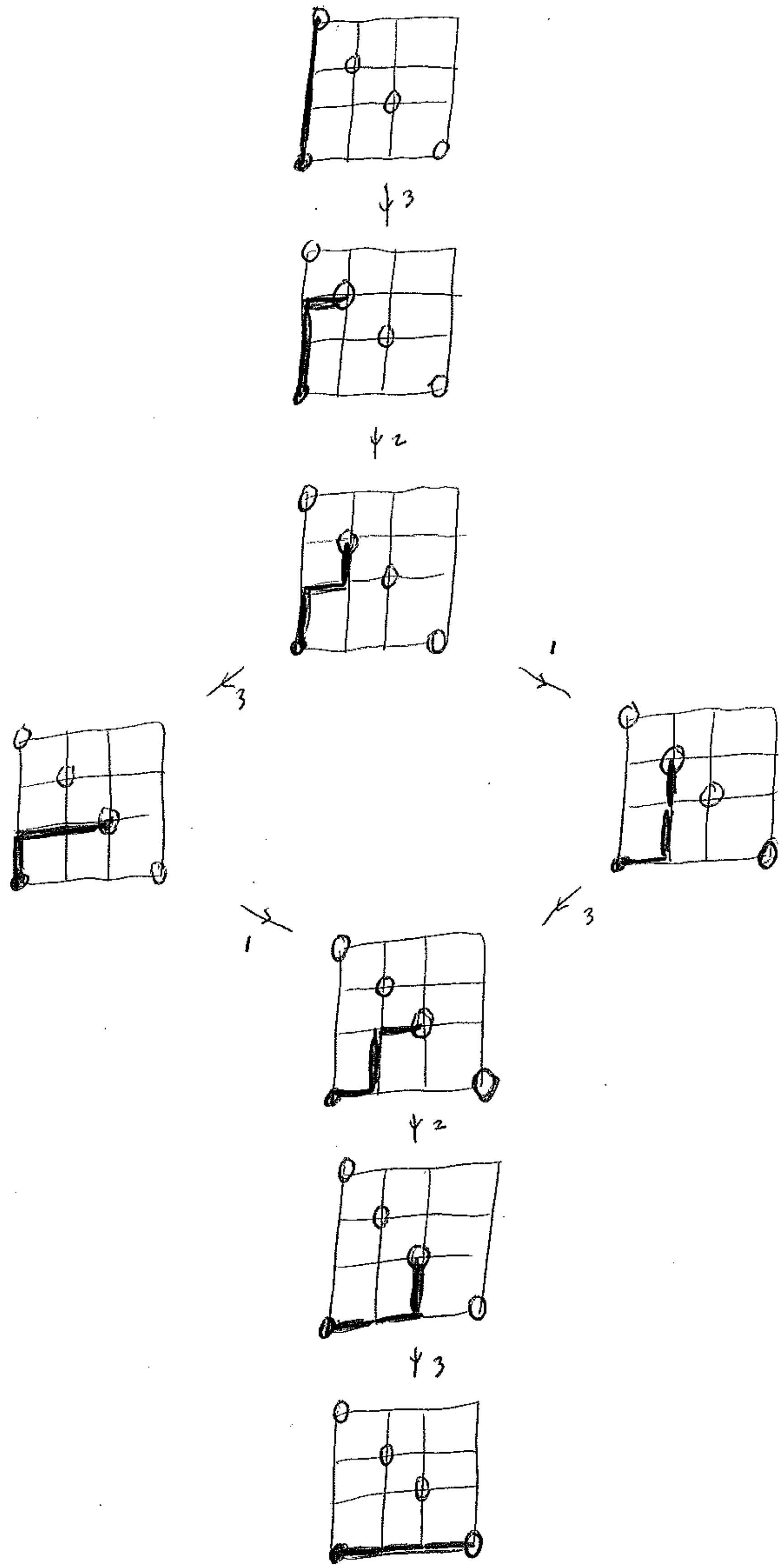
$x \xleftarrow[i]{} y$ whenever x is obtained from y

by "adding" an i-block.

$r = 3$ revisited

3		
2	3	
1	2	3

$\Phi = B_r$ $\quad \lambda = \varpi_r$ $\quad M_\lambda$ $\quad$ for $r=3$



Ex $\quad \Phi = D_r \qquad \lambda = \varpi_{r-1} = \dfrac{e_1 + \cdots + e_{r-1} - e_r}{2}$

$$M_\lambda = \left\{ \frac{1}{2} \sum_{i=1}^{r} \varepsilon_i e_i \;\middle|\; \varepsilon_i \in \{+,-\},\ |\{i \mid \varepsilon_i = -\}| \text{ odd} \right\}$$

Fa $1 \le i \le r-1$

$$\underset{i\ \ i+1}{** \cdots * - + * \cdots **} \xleftarrow{\quad i \quad} \underset{i\ \ i+1}{** \cdots * + - * \cdots **}$$

$*$ arb

$$** \cdots * - - \xleftarrow{\quad r \quad} ** \cdots * + +$$

$\Phi = D_r \qquad \lambda = \overline{\omega_1} \qquad M_\lambda \qquad r = 4$

$$\begin{tikzcd}
 & {+++-} \arrow[d, "3"] & \\
 & {++-+} \arrow[d, "2"] & \\
 & {+-++} \arrow[dl, "1"'] \arrow[dr, "4"] & \\
{-+++} \arrow[dr, "4"'] & & {+---} \arrow[dl, "1"] \\
 & {-+--} \arrow[d, "2"] & \\
 & {--+-} \arrow[d, "3"] & \\
 & {---+} &
\end{tikzcd}$$

$\Phi = D_r$ $\lambda = \overline{\omega}_r$ M_λ $r = 5$

$$
\begin{tikzcd}
 & & \text{++++-} \arrow[d, "4"] & & \\
 & & \text{+++-+} \arrow[d, "3"] & & \\
 & & \text{++-++} \arrow[dl, "2"'] \arrow[dr, "5"] & & \\
 & \text{+-+++} \arrow[dl, "1"'] \arrow[dr, "5"] & & \text{++---} \arrow[dl, "2"] & \\
\text{-++++} \arrow[dr, "5"] & & \text{+-+--} \arrow[dl, "1"'] \arrow[dr, "3"] & & \\
 & \text{-++--} \arrow[dr, "3"'] & & \text{+--+-} \arrow[dl, "1"'] \arrow[dr, "4"] & \\
 & & \text{-+-+-} \arrow[dl, "2"'] \arrow[dr, "4"] & & \text{+---+} \arrow[dl, "1"] \\
 & \text{--++-} \arrow[dr, "4"] & & \text{-+--+} \arrow[dl, "2"] & \\
 & & \text{--+-+} \arrow[d, "3"] & & \\
 & & \text{---++} \arrow[d, "5"] & & \\
 & & \text{-----} & &
\end{tikzcd}
$$

Compare:

- M_λ for B_r $\quad \lambda = \overline{\omega}_r$
- M_λ for D_{r+1} $\quad \lambda = \overline{\omega}_r$

<u>Notation</u>

For $\varepsilon \in \{+, -\}$

define $\overline{\varepsilon} = \begin{cases} + & \text{if } \varepsilon = - \\ - & \text{if } \varepsilon = + \end{cases}$

For $x \in M_\lambda(B_r)$

write $x = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_r$

define $s(x) = \begin{cases} + & \text{if even number of } \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \text{ are } - \\ - & \text{if odd number of } \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \text{ are } - \end{cases}$

$\exists$ bijection

$$\theta : \begin{array}{ccc} M_\lambda(B_r) & \longrightarrow & M_\lambda(D_{r+1}) \\ x & \longrightarrow & x\,\overline{s(x)} \end{array}$$

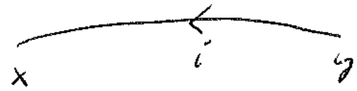
So for $r=3$, θ sends

$$++- \longrightarrow ++-+$$

$$--+ \longrightarrow --+-$$

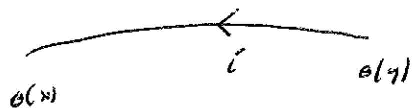
Given $x, y \in M_\lambda(B_r)$

For $1 \le i \le r-1$



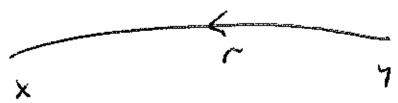
in $M_\lambda(B_r)$

iff



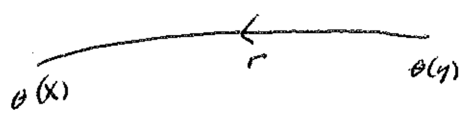
in $M_\lambda(D_{r+1})$

Also



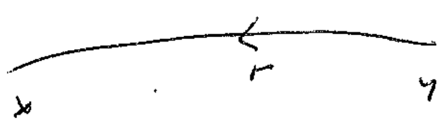
in $M_\lambda(B_r)$ and $s(y) = +$

iff



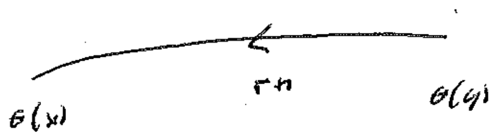
in $M_\lambda(D_{r+1})$

Also



in $M_\lambda(B_r)$ and $s(y) = -$

iff



in $M_\lambda(D_{r+1})$