Lecture 22 Wednesday Oct 23

Find $\hat{\varphi}_j(b \otimes c)$, $\hat{\varepsilon}_j(b \otimes c)$

(view I) Using signature rule

$$b \otimes c$$

$$\underbrace{))\cdots)}_{\hat{\varphi}_j(b)}\ \underbrace{(\cdots((}_{\hat{\varepsilon}_j(b)} \qquad \underbrace{))\cdots)}_{\hat{\varphi}_j(c)}\ \underbrace{(\cdots((}_{\hat{\varepsilon}_j(c)}$$

cancel (,) to get

$$\underbrace{))\cdots)}_{\hat{\varphi}_j(b\otimes c)}\ \underbrace{(\cdots((}_{\hat{\varepsilon}_j(b\otimes c)} \qquad \star$$

(view II) repeating,

$$b \otimes c$$

$$\underbrace{))\cdots)}_{\hat{\varphi}_j(b)\,=\,\gamma_i\varphi_i(b)}\ \underbrace{(\cdots((}_{\hat{\varepsilon}_j(b)\,=\,\gamma_i\varepsilon_i(b)} \qquad \underbrace{))\cdots)}_{\hat{\varphi}_j(c)\,=\,\gamma_i\varphi_i(c)}\ \underbrace{(\cdots((}_{\hat{\varepsilon}_j(c)\,=\,\gamma_i\varepsilon_i(c)}$$

cancel (,) "in groups of γ_i" to get

$$\underbrace{))\cdots)}_{\gamma_i\varphi_i(b\otimes c)}\ \underbrace{(\cdots((}_{\gamma_i\varepsilon_i(b\otimes c)} \qquad \star\star$$

But ☆, ☆☆ are same so

$$\hat{\varphi}_j(b \otimes c) = \gamma_i \varphi_i(b \otimes c),$$

$$\hat{\varepsilon}_j(b \otimes c) = \gamma_i \varepsilon_i(b \otimes c)$$

V3: $\mathcal{B}, \mathcal{C}$ are crystals

so $\mathcal{B} \otimes \mathcal{C}$ is crystal

Corresp. functions written

$E_i, \quad F_i \qquad i \in I^X$

Crystals $\mathcal{B}, \mathcal{C}$ are SN so

$\mathcal{B} \otimes \mathcal{C}$ is SN.

Also have virtual operators for crystal $\hat{\mathcal{B}} \otimes \hat{\mathcal{C}}$:

$e_i, \quad f_i \qquad i \in I^X$

Show that on $\mathcal{B} \otimes \mathcal{C}$

$e_i = E_i \qquad f_i = F_i \qquad i \in I^X$

↑

Either

$$|\sigma(i)| = 1 \quad \text{or} \quad \delta_i = 1$$

Assume $|\sigma(i)| = 1$

Write $\sigma(i) = \{j\}$

So $f_i = \hat{f}_j^{\,\delta_i}$

Find $F_i(b \otimes c)$

sign rule

$$\begin{array}{ccc} b & \otimes & c \\ ) \cdots) \; (\cdots (& &) \cdots) \; (\cdots (\\ \varphi_i(b) \;\; \varepsilon_i(b) & & \varphi_i(c) \;\; \varepsilon_i(c) \end{array}$$

cancel (,) to get

$$) \cdots) \; (\cdots ($$

<u>Case I</u> $\ast$ is

$$b \otimes c \qquad \ast\ast$$

$$> \cdots > < \cdots < \qquad < \cdots <$$

$$\uparrow$$

$$F_i(b \otimes c) = F_i(b) \otimes c$$
$$= \hat{f}_j^{\gamma_i}(b) \otimes c$$

<u>Case II</u> $\ast$ is

$$b \otimes c \qquad \ast\ast\ast$$

$$> \cdots > \qquad > \cdots > < \cdots <$$

$$\uparrow$$

$$F_i(b \otimes c) = b \otimes F_i(c)$$
$$= b \otimes \hat{f}_j^{\gamma_i}(c)$$

Find $f_i(b \otimes c)$

Apply $\hat{f}_j^{\,\gamma_i}$ to $b \otimes c$ γ_i times

sign rule

$$b \otimes c$$

$$\underbrace{) \cdots) \; (\cdots (}_{} \quad) \cdots) \; (\cdots ($$

$$\hat{\varphi}_j(b) \quad \hat{\varepsilon}_j(b) \quad \hat{\varphi}_j(c) \quad \hat{\varepsilon}_j(c)$$

$$\| \quad \| \quad \| \quad \|$$

$$\gamma_i \varphi_i(b) \quad \gamma_i \varepsilon_i(b) \quad \gamma_i \varphi_i(c) \quad \gamma_i \varepsilon_i(c)$$

cancel $(\,)$ to get

$$) \cdots) \; (\cdots ($$

(obtained from * by replicating each $), ($ γ_i times)

Either

Case I

$$b \otimes c$$

$$) \cdots) \; (\cdots (\quad (\cdots ($$

$$\uparrow$$

(obtained from ** by replicating each $), ($ γ_i times)

$$f_i(b \otimes c) = \hat{f}_j^{\,\gamma_i}(b) \otimes c$$

or

Case II

$$b \otimes C$$

$$\gamma \cdots \gamma \quad \gamma \cdots \gamma\, C \cdots C$$

$$\uparrow$$

(obtained from γ^{**} by replicating each $\gamma_i C$ γ_i times)

$$f_i(b \otimes c) = b \otimes \hat{f}_j^{\gamma_i}(c)$$

In each case

$$f_i = F_i$$

For $\gamma_i = 1$ we have

$$f_i = \prod_{j \in \sigma(i)} \hat{f}_j$$

and the argument is similar. (ex)

$\square$

LEM Given virtual crystals

$B \subseteq \hat{B}$

$C \subseteq \hat{C}$

X Y

with B, C connected.

[so B, C have unique hw]

Assume hw of B = hw of C $= \lambda$

then crystals B, C are iso.

pf $B \subseteq$ connected comp of $\hat{B}$

$C \subseteq$ -- $\hat{C}$

WLOG $\hat{B}, \hat{C}$ connected

hw element u of B is hw in $\hat{B}$ $*$

B is obtained by applying virtual ops of $\hat{B}$ to u

hw element v of C is hw in $\hat{C}$

C is obtained by applying virtual ops of $\hat{C}$ to v $**$

$$\text{hwg}\,\hat{B} = \hat{wt}(u) = \Psi(\lambda) = \hat{wt}(v) = \text{hwg}\,\hat{C}$$

crystals $\hat{B}$, $\hat{C}$ are iso

$\exists$ crystal iso $\gamma : \hat{B} \to \hat{C}$

By cnstr $\gamma(u) = v$

WLOG identify $\hat{B}$, $\hat{C}$ via γ

so $u = v$

For $i \in I^{\times}$,

$\hat{B}$, $\hat{C}$ give same virtual ops e_i

and same virtual ops f_i

Now by $\times$, $\times\times$

$$B = C$$

$\square$

Result follows.

Motivation

For $\Phi = A_r$, $GL(r+1)$

and for each partition $\lambda \in \Lambda^+$ we defined a crystal

$\mathcal{B}_\lambda$. $\mathcal{B}_\lambda$ is Stembridge, connected, has unique hw

vector, hw λ

Next, consider any other root system Φ from the classification:

$B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2$

Take $\Lambda = \Lambda_{sc}$

For $\lambda \in \Lambda^+$ we wish to define a certain connected

Seminormal crystal $\mathcal{B}_\lambda$ with a unique hw vector and hw λ

To do this, we first define $\mathcal{B}_\lambda$ assuming $\lambda = \varpi_k$ is a

fundamental dominant wt. Such a $\mathcal{B}_\lambda$ is called a

fundamental crystal

For $\Phi = B_r, C_r, D_r$ consider the standard crystal

$$\mathbb{B} = B_{\varpi_1}$$

For $1 \le k \le r$ consider

$$u = \boxed{k} \otimes \boxed{k\text{-}1} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$$

Note

$$wt(u) = e_1 + e_2 + \cdots + e_k$$

LEM With above notation,

(i) u is hw vector in the crystal $\mathbb{B}^{\otimes k}$

(ii) For $\Phi = B_r$,

$$wt(u) = \begin{cases} \varpi_k & \text{if } 1 \le k \le r-1 \\ 2\varpi_r & \text{if } k = r \end{cases}$$

(iii) For $\Phi = C_r$,

$$wt(u) = \varpi_k \qquad 1 \le k \le r$$

(iv) For $\Phi = D_r$

$$wt(u) = \begin{cases} \varpi_k & \text{if } 1 \le k \le r-2 \\ \varpi_{r-1} + \varpi_r & \text{if } k = r-1 \\ 2\varpi_r & \text{if } k = r \end{cases}$$

pf (i) Routinely check $\varepsilon_i(u) = 0$ for $1 \le i \le r$ using signature rule

(ii)–(iv) Use formula for ϖ_k

$\square$

LEM With above notation, let

C = the connected component of the crystal $\mathbb{B}^{\otimes k}$

that contains u.

Then

(i) For $\Phi = B_r, C_r$

C is a virtual crystal

(ii) For $\Phi = D_r$

C is Stembridge

(iii) u is unique hw vector in C

pf (i) $\mathbb{B}$ is virtual so $\mathbb{B}^{\otimes k}$ is virtual

(ii) $\mathbb{B}$ is Stembridge so $\mathbb{B}^{\otimes k}$ is Stembridge

(iii) By (i), (ii) $\square$

For $\Phi = D_r$

consider $v = \boxed{\bar{r}} \otimes \boxed{r-1} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1} \quad \in \mathbb{B}^{\otimes r}$

Note
$wt(v) = e_1 + e_2 + \cdots + e_{r-1} - e_r$

LEM With above notation

(i) v is hw vector in crystal $\mathbb{B}^{\otimes r}$

(ii) $wt(v) = 2\bar{\omega}_{r-1}$

pf Routine $\square$

LEM With above notation, let

C = the connected component of the crystal $\mathbb{B}^{\otimes r}$ that contains v.

then (i) C is Stembridge

(ii) v is unique hw vector in C

pf (i) $\mathbb{B}$ is Stembridge so $\mathbb{B}^{\otimes r}$ is Stembridge

(ii) By (i) $\square$