

Find $\hat{\varphi}_2(b \otimes c), \hat{\varepsilon}_2(b \otimes c)$

(view I) Using signature rule

$$\begin{array}{ccc}
 b & \otimes & c \\
 \underbrace{\rightsquigarrow \dots \rightsquigarrow} & & \underbrace{\rightsquigarrow \dots \rightsquigarrow} \\
 \hat{\varphi}_2(b) & \hat{\varepsilon}_2(b) & \hat{\varphi}_2(c) \quad \hat{\varepsilon}_2(c)
 \end{array}$$

cancel (.) to get

$$\underbrace{\rightsquigarrow \dots \rightsquigarrow} \quad \underbrace{\rightsquigarrow \dots \rightsquigarrow} \\
 \hat{\varphi}_2(b \otimes c) \quad \hat{\varepsilon}_2(b \otimes c)$$

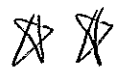


(view II) repeating,

$$\begin{array}{ccc}
 b & \otimes & c \\
 \underbrace{\rightsquigarrow \dots \rightsquigarrow} & & \underbrace{\rightsquigarrow \dots \rightsquigarrow} \\
 \hat{\varphi}_2(b) & \hat{\varepsilon}_2(b) & \hat{\varphi}_2(c) \quad \hat{\varepsilon}_2(c) \\
 \parallel & \parallel & \parallel \quad \parallel \\
 \gamma_i \varphi_i(b) & \gamma_i \varepsilon_i(b) & \gamma_i \varphi_i(c) \quad \gamma_i \varepsilon_i(c)
 \end{array}$$

cancel (.) " in groups of γ_i " to get

$$\underbrace{\rightsquigarrow \dots \rightsquigarrow} \quad \underbrace{\rightsquigarrow \dots \rightsquigarrow} \\
 \gamma_i \varphi_i(b \otimes c) \quad \gamma_i \varepsilon_i(b \otimes c)$$



But $\star, \star\star$ are same so

$$\hat{\varphi}_2(b \otimes c) = \gamma_i \varphi_i(b \otimes c),$$

$$\hat{\varepsilon}_2(b \otimes c) = \gamma_i \varepsilon_i(b \otimes c)$$

V3: B, C are crystals

so $B \otimes C$ is crystal

Corresp functions written

$$E_i, F_i \quad i \in I^x$$

Crystals B, C are SN so

$B \otimes C$ is SN.

Also have virtual operators for crystal $\hat{B} \otimes \hat{C}$:

$$e_i, f_i \quad i \in I^x$$

Show that on $B \otimes C$

$$e_i = \hat{E}_i$$

$$f_i = \hat{F}_i$$

$$i \in I^x$$

↑

Either

$$|\sigma(i)| = 1 \quad \text{or} \quad \delta_i = 1$$

Assume $|\sigma(i)| = 1$

Write $\sigma(i) = \{j\}$

So $f_i = f_j^{\delta_i}$

Find $F_i(b \otimes c)$

Sign rule

$$\begin{array}{ccc}
 b & \otimes & c \\
 \nearrow \dots \rightarrow c \dots c & & \nearrow \dots \rightarrow c \dots c \\
 \psi_i(b) \quad \varepsilon_i(b) & & \psi_i(c) \quad \varepsilon_i(c)
 \end{array}$$

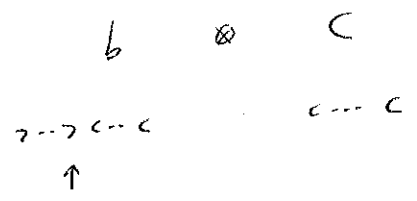
cancel (i) to get

$$\begin{array}{ccc}
 \nearrow \dots \rightarrow c \dots c \\
 \uparrow
 \end{array}$$

*

Case I

* IS

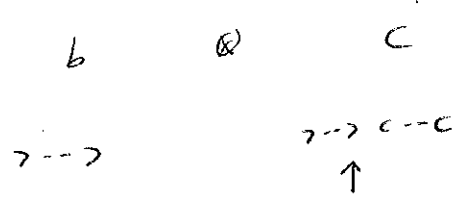


**

$$\begin{aligned}
 F_i(b \otimes C) &= F_i(b) \otimes C \\
 &= \hat{F}_i^{X_i}(b) \otimes C
 \end{aligned}$$

Case II

* IS



$$\begin{aligned}
 F_i(b \otimes C) &= b \otimes F_i(C) \\
 &= b \otimes \hat{F}_i^{X_i}(C)
 \end{aligned}$$

Find $f_i(b \otimes c)$

Apply $\hat{f}_i^{\gamma_i}$ to $b \otimes c$ γ_i times

sign rule

$$b \otimes c$$

$$\begin{array}{cccc}
 \gamma \dots \gamma & \gamma \dots \gamma & \gamma \dots \gamma & \gamma \dots \gamma \\
 \hat{\varphi}_i(b) & \hat{\varphi}_i(b) & \hat{\varphi}_i(c) & \hat{\varphi}_i(c) \\
 \text{"} & \text{"} & \text{"} & \text{"} \\
 \gamma_i \varphi_i(b) & \gamma_i \varphi_i(b) & \gamma_i \varphi_i(c) & \gamma_i \varphi_i(c)
 \end{array}$$

cancel (,) to get

$$\gamma \dots \gamma \quad c \dots c$$

(obtained from \ast
by replicating each γ, c
 γ_i times)

Either

Case I

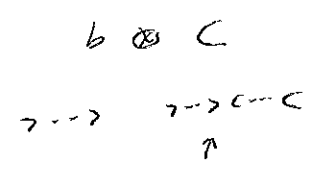
$$\begin{array}{ccc}
 b & \otimes & c \\
 \gamma \dots \gamma & & c \dots c \\
 \uparrow & & \\
 & &
 \end{array}$$

(obtained from $\ast\ast$
by replicating each γ, c
 γ_i times)

$$f_i(b \otimes c) = \hat{f}_i^{\gamma_i}(b) \otimes c$$

or

Case II



(obtained from ~~***~~
 by replicating each $\gamma_i C$
 γ_i times)

$$f_i(b \otimes C) = b \otimes f_i^{\gamma_i}(C)$$

In each case

$$f_i = F_i$$

For $\gamma_i \neq 1$ we have

$$f_i = \prod_{\gamma \in \sigma(i)} \hat{f}_i$$

and the argument is similar. (ex)



LEM Given virtual crystals

$$B \subseteq \hat{B}$$

$$C \subseteq \hat{C}$$

$$X \cdot Y$$

with B, C connected.

[so B, C have unique hw]

Assume

$$\text{hw of } B = \text{hw of } C = \lambda$$

then crystals B, C are iso.

pf $B \subseteq$ connected comp of \hat{B}
 $C \subseteq$ -- \hat{C}

wlog \hat{B}, \hat{C} connected

hw element u of B is hw in \hat{B}

B is obtained by applying virtual ops of \hat{B} to u *

hw element v of C is hw in \hat{C}

C is obtained by applying virtual ops of \hat{C} to v **

$$\text{hw of } \hat{B} = \text{wt}(u) = \psi(\lambda) = \text{wt}(v) = \text{hw of } \hat{C}$$

crystals \hat{B}, \hat{C} are iso

\exists crystal iso $\gamma: \hat{B} \rightarrow \hat{C}$

By const $\gamma(u) = v$

WLOG identify \hat{B}, \hat{C} via γ
so $u = v$

For $i \in I^X$,

\hat{B}, \hat{C} give same virtual ops e_i
and same virtual ops f_i

Now by $K_1 X$

$$\hat{B} = \hat{C}$$

Result follows.



Motivation

For $\Phi = A_n, GL(m)$
 and for each partition $\lambda \in \Lambda^+$ we defined a crystal B_λ .
 B_λ is skeinbridge, connected, has unique h.w. vector, h.w. λ

Next, consider any other root system Φ from the classification:

- $B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$

Take $\Lambda = \Lambda_{sc}$

For $\lambda \in \Lambda^+$ we wish to define a certain connected seminormal crystal B_λ with a unique h.w. vector and h.w. λ

To do this, we first define B_λ assuming $\lambda = \bar{w}_k$ is a fundamental dominant wt. Such a B_λ is called a

fundamental crystal

For $\Phi = B_r, C_r, D_r$ consider the standard crystal

$$B = B\bar{w}_1$$

For $1 \leq k \leq r$ consider

$$u = \boxed{k} \otimes \boxed{k+1} \otimes \dots \otimes \boxed{2} \otimes \boxed{1} \in B^{\otimes k}$$

Note

$$\text{wt}(u) = e_1 + e_2 + \dots + e_k$$

LEM With above notation,

(i) u is h.w. vector in the crystal $B^{\otimes k}$

(ii) For $\Phi = B_r$,

$$\text{wt}(u) = \begin{cases} \bar{w}_k & \text{if } 1 \leq k \leq r-1 \\ 2\bar{w}_r & \text{if } k=r \end{cases}$$

(iii) For $\Phi = C_r$,

$$\text{wt}(u) = \bar{w}_k \quad 1 \leq k \leq r$$

(iv) For $\Phi = D_r$

$$\text{wt}(u) = \begin{cases} \bar{w}_k & \text{if } 1 \leq k \leq r-2 \\ \bar{w}_{r-1} + \bar{w}_r & \text{if } k=r-1 \\ 2\bar{w}_r & \text{if } k=r \end{cases}$$

pf (i) Routinely check $\epsilon_i(u) = 0$ for $1 \leq i \leq r$ using signature rule

(ii)-(iv) Use formula for \bar{w}_k

□

LEM With above notation, let

C = the connected component of the crystal $B^{\otimes k}$ that contains u .

Then

(i) For $\Phi = B_r, C_r$

C is a virtual crystal

(ii) For $\Phi = D_r$

C is Stembridge

(iii) u is unique hw vector in C

pf (i) B is virtual so $B^{\otimes k}$ is virtual

(ii) B is Stembridge so $B^{\otimes k}$ is Stembridge

(iii) By (i), (ii)

□

For $\mathbb{F} = \mathbb{D}_r$

consider

$$v = \boxed{r} \otimes \boxed{r-1} \otimes \dots \otimes \boxed{2} \otimes \boxed{1} \in \mathbb{B}^{\otimes r}$$

Note

$$\text{wt}(v) = e_1 + e_2 + \dots + e_{r-1} - e_r$$

LEM With above notation

(i) v is h.w. vector in crystal $\mathbb{B}^{\otimes r}$

(ii) $\text{wt}(v) = 2\bar{w}_{r-1}$

pf Routine

□

LEM With above notation, let

$C =$ the connected component of the crystal $B^{\otimes r}$
that contains v .

then (i) C is Stembridge

(ii) v is unique hw vector in C

pf (i) B is Stembridge so $B^{\otimes r}$ is Stembridge

(ii) by (i)

□