10/21/19
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Lecture 21 Monday Oct 21

Properties of virtual crystals.

Given a virtual crystal

$$B \subseteq \hat{B}$$

X system Y system

LEM Assume $u \in B$ is hw. Then

(i) $u \in \hat{B}$ is hw [write $\lambda = \hat{wt}(u)$]

(ii) λ is fixed by diagram aut

pf (i) u is aligned in $\hat{B}$

For $j \in I^Y$ show

$$\hat{\varepsilon}_j(u) = 0$$

$\exists\ i \in I^X$ st

$$j \in \sigma(i)$$

Recall

$$\underset{\substack{\| \\ 0}}{\varepsilon_i(u)} = \frac{\hat{\varepsilon}_j(u)}{\gamma_i}$$

since u is hw in B

So $\hat{\varepsilon}_j(u) = 0$

Pf (ii) Write

$$wt(u) = \sum_{i \in I^X} a_i \bar{\omega}_i^X \qquad a_i \in \mathbb{Z}$$

Apply ψ:

$$\psi(wt(u)) = \sum_{i \in I^X} a_i \, \psi(\bar{\omega}_i^X)$$

$$\begin{aligned} &\parallel \\ &\hat{wt}(u) \\ &\parallel \\ &\lambda \end{aligned}$$

Fn $i \in I^X$

$$\psi(\bar{\omega}_i^X) = \gamma_i \sum_{j \in \sigma(i)} \bar{\omega}_j^Y$$

aut induces gp iso $\Lambda^Y \to \Lambda^Y$ that leaves Φ^Y invar.

aut permutes $\{\alpha_j^Y\}_{j \in I^Y}$

... $\{\bar{\omega}_j^Y\}_{j \in I^Y}$

Fn $i \in I^X$, aut permutes $\{\bar{\omega}_j^Y\}_{j \in \sigma(i)}$

So aut leaves $\psi(\bar{\omega}_i^X)$ invar $\forall i \in I^X$

Result follows. $\square$

LEM For above B, $\hat{B}$

assume B is connected. then

B has a unique hw element.

pf. B is contained in connected components of $\hat{B}$

WLOG $\hat{B}$ is connected

Stembridge crystal $\hat{B}$ has unique hw element u.

Each hw element of B is hw in $\hat{B}$, so is equal to u. $\square$

LEM Assume above crystals $B, \hat{B}$ are connected.

Then $\exists$ bijection

$$\mathrm{aut}: \hat{B} \to \hat{B}$$

such that

(i) For $x \in \hat{B}$,

$$\hat{wt}(\mathrm{aut}(x)) = \mathrm{aut}(\hat{wt}(x))$$

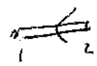
(ii) For $x, y \in \hat{B}$ and $i \in I^Y$,

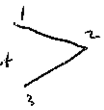
$$x \xleftarrow{\;i\;} y \quad \text{iff} \quad \mathrm{aut}(x) \xleftarrow{\;\mathrm{aut}(i)\;} \mathrm{aut}(y)$$

(iii) For $x \in \hat{B}$ and $i \in I^Y$

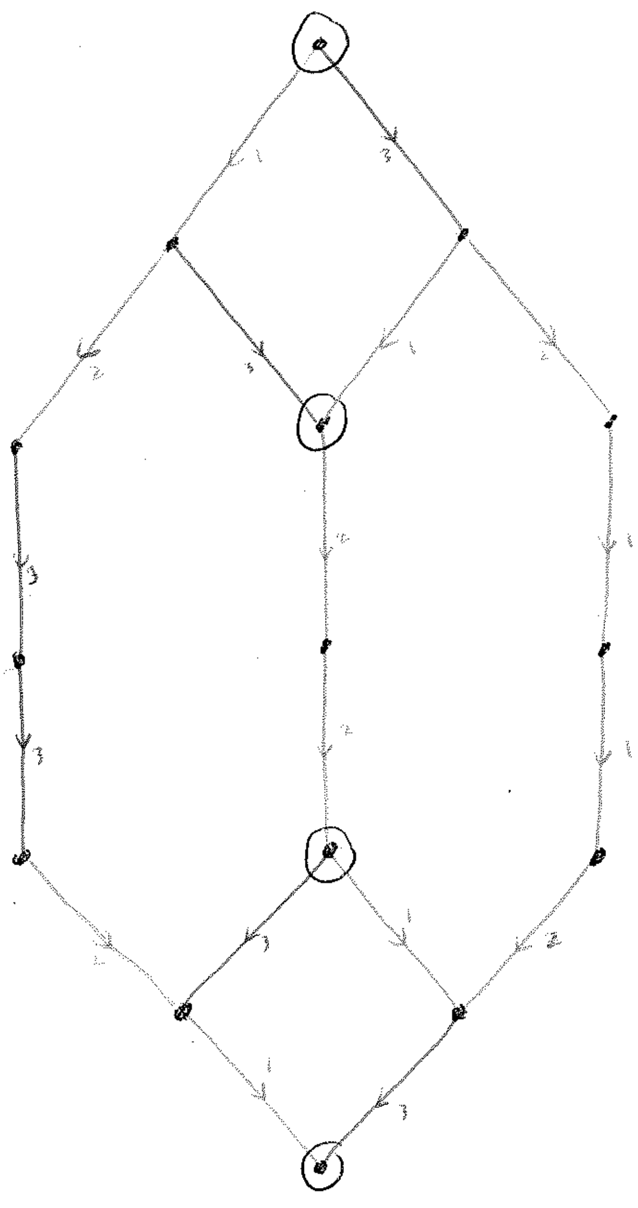
$$\varphi_{\mathrm{aut}(i)}(\mathrm{aut}(x)) = \varphi_i(x), \qquad \varepsilon_{\mathrm{aut}(i)}(\mathrm{aut}(x)) = \varepsilon_i(x)$$

Moreover aut fixes each element of B.

[ex C_2  X

A_3 $\updownarrow$ aut  Y

aut swaps 



aut

circled nodes are in B

f_1 f_2 f_3]

Pf of LEM

Connected Stembridge crystal $\hat{B}$ has unique hw element u.

B has hw element, which is hw in $\hat{B}$, so is equal to u.

So $u \in B$

u is aligned

$\lambda = \hat{wt}(u)$ is fixed by aut.

By constr

$$\langle \mathrm{aut}(x), \mathrm{aut}(y) \rangle = \langle x, y \rangle \qquad x, y \in \Lambda^Y$$

Using aut we constr a new crystal str on $\hat{B}$ as follows.

- $\forall x, y \in \hat{B} \quad \forall i \in I^Y$

$$x \xleftarrow{\;i\;} y \quad \text{iff} \quad x \xleftarrow{\;\mathrm{aut}(i)\;} y$$

new

- $\forall x \in \hat{B}$,

$$\hat{wt}^{\,new}(x) = \mathrm{aut}^{-1}\left(\hat{wt}(x)\right)$$

• $\forall x \in \hat{B}$ $\quad \forall i \in I^Y$

$$\varphi_i^{new}(x) = \varphi_{aut(i)}(x),$$

$$\varepsilon_i^{new}(x) = \varepsilon_{aut(i)}(x)$$

Check new $\hat{B}$ really is crystal:

Given $x, y \in \hat{B}$ and $i \in I^Y$ st

$$x \xleftarrow[\text{new}]{i} y$$

Require

$$\underset{\substack{\| \\ aut^{-1}\left(\hat{wt}(y)\right)}}{\hat{wt}^{new}(y)} - \underset{\substack{\| \\ aut^{-1}\left(\hat{wt}(x)\right)}}{\hat{wt}^{new}(x)} \overset{?}{=} \alpha_i$$

$$\hat{wt}(y) - \hat{wt}(x) \overset{?}{=} \underset{\substack{\| \\ \alpha_{aut(i)}}}{aut(\alpha_i)}$$

ok since

$$x \xleftarrow{aut(i)} y$$

For $x \in \hat{B}$ and $i \in I^Y$,

require

$$\varphi_i^{new}(x) - \varepsilon_i^{new}(x) \overset{?}{=} \left\langle \hat{wt}^{new}(x), \alpha_i^\vee \right\rangle$$

$\varphi_i^{new}(x) = \varphi_{aut(i)}(x)$, $\quad \varepsilon_i^{new}(x) = \varepsilon_{aut(i)}(x)$, $\quad \hat{wt}^{new}(x) = aut^{-1}(\hat{wt}(x))$

$$\underbrace{\left\langle aut^{-1}(\hat{wt}(x)), \alpha_i^\vee \right\rangle} = \left\langle \hat{wt}(x), aut(\alpha_i^\vee) \right\rangle$$

$$\varphi_{aut(i)}(x) - \varepsilon_{aut(i)}(x) \underset{ok}{=} \left\langle \hat{wt}(x), \alpha^\vee_{aut(i)} \right\rangle$$

So $\hat{B}^{new}$ is crystal.

By constr $\hat{B}^{new}$ is connected, Stembridge.

Recall hw vector u for $\hat{B}$, with hw λ

show u is hw in $\hat{B}^{new}$:

For $i \in I^Y$,

$$\varepsilon_i^{new}(u) = \varepsilon_{aut(i)}(u) = 0$$

Find wt of u in $\hat{B}^{new}$:

$$\hat{wt}^{new}(u) = aut^{-1}(\hat{wt}(u))$$

$$= aut^{-1}(\lambda)$$

$$= \lambda.$$

Crystals $\hat{B}$, $\hat{B}^{new}$ have same hw λ,

so $\hat{B}$, $\hat{B}^{new}$ are iso

$\exists$ crystal iso

$$\text{aut} : \hat{B} \longrightarrow \hat{B}^{new}$$

then $\text{aut} : \hat{B} \to \hat{B}$ is a bij that satisfies (i) – (iii)

show aut fixes everything in B.

We saw $u \in B$

u is hw element for $\hat{B}$, $\hat{B}^{new}$

so $\text{aut}(u) = u$

Since B is connected, suf to show:

$\forall$ aligned elements $x, y \in \hat{B}$ and $i \in I^{x}$,

$$x \xleftarrow{\;i\;} y \quad \text{and} \quad \text{aut}(y) = y$$

implies $\text{aut}(x) = x$.

This is routinely checked using the def of aligned. $\square$

LEM Given virtual crystals

$$B \subseteq \hat{B}$$

$$C \subseteq \hat{C}$$

$$X \quad Y$$

then we have virtual crystal

$$B \otimes C \subseteq \hat{B} \otimes \hat{C}$$

$$X \quad Y$$

pf check axioms V1, V2, V3

V1 : $\hat{B}$, $\hat{C}$ are Stembridge so $\hat{B} \otimes \hat{C}$ is Stembridge

V2 : For $b \otimes c \in B \otimes C$ and $i \in I^X$

show

$$\varphi_i(b \otimes c) = \frac{\hat{\varphi}_j(b \otimes c)}{\gamma_i} \qquad j \in \sigma(i)$$

$$\varepsilon_i(b \otimes c) = \frac{\hat{\varepsilon}_j(b \otimes c)}{\gamma_i} \qquad ''$$

Since b, c are aligned,

$$\varphi_i(b) = \frac{\hat{\varphi}_j(b)}{\gamma_i}$$

$$\varepsilon_i(b) = \frac{\hat{\varepsilon}_j(b)}{\gamma_i}$$

$$\varphi_i(c) = \frac{\hat{\varphi}_j(c)}{\gamma_i}$$

$$\varepsilon_i(c) = \frac{\hat{\varepsilon}_j(c)}{\gamma_i}$$

Find $\varphi_i(b \otimes c)$, $\varepsilon_i(b \otimes c)$

using signature rule

$$b \otimes c$$

$$\underbrace{)) \cdots)}_{\varphi_i(b)} \underbrace{(\cdots ((}_{\varepsilon_i(b)} \quad \underbrace{)) \cdots)}_{\varphi_i(c)} \underbrace{(\cdots ((}_{\varepsilon_i(c)}$$

Cancel $(\,)$ to get

$$\underbrace{)) \cdots)}_{\varphi_i(b \otimes c)} \quad \underbrace{(\cdots (\, (}_{\varepsilon_i(b \otimes c)}$$