Lecture 20 Friday Oct 18 10/18/19

Describe the virtual operators on the aligned elements 1

	$\widehat{wt}$	wt
$\boxed{1} \otimes \boxed{1}$	$2e_1$	e_1
$\downarrow 1$		
$\boxed{2} \otimes \boxed{2}$	$2e_2$	e_2
$\downarrow 2$		
$\boxed{2} \otimes \boxed{\bar{2}}$	0	0
$\downarrow 2$		
$\boxed{\bar{2}} \otimes \boxed{\bar{2}}$	$-2e_2$	$-e_2$
$\downarrow 1$		
$\boxed{\bar{1}} \otimes \boxed{\bar{1}}$	$-2e_1$	$-e_1$

This is standard crystal for B_2

hw is $e_1 = \varpi_1$

Call it $\mathcal{B}_{\varpi_1}$ for B_2

Describe virtual operators on the aligned elements, cont.

$$\begin{array}{lcc}
 & \widehat{wt} & wt \\
\boxed{2} \otimes \boxed{1} & e_1+e_2 & \frac{e_1+e_2}{2} \\
\downarrow 2 & & \\
\boxed{\bar{2}} \otimes \boxed{1} & e_1-e_2 & \frac{e_1-e_2}{2} \\
\downarrow 1 & & \\
\boxed{\bar{1}} \otimes \boxed{2} & e_2-e_1 & \frac{e_2-e_1}{2} \\
\downarrow 2 & & \\
\boxed{\bar{1}} \otimes \boxed{\bar{2}} & -e_1-e_2 & -\frac{e_1+e_2}{2}
\end{array}$$

This is B_2-crystal with hw $\frac{e_1+e_2}{2} = \varpi_2$

call it B_{ϖ_2} for B_2

Also

$$\begin{array}{ccc} & \hat{wt} & wt \\ \boxed{\bar{1}} \otimes \boxed{1} & 0 & 0 \end{array}$$

is B_2-crystal

We have now realized the crystals

B_{ϖ_1}, B_{ϖ_2} for B_2

as virtual crystals for the D_3-crystal $B_{\varpi_1} \otimes B_{\varpi_1}$

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Next goal: For $r \geq 2$ realize

B_r-crystal B_{ϖ_1} as a virtual crystal

in the D_{r+1}-crystal $B_{\varpi_1} \otimes B_{\varpi_1}$

D_{r+1}-crystal

$$
\begin{tikzcd}[column sep=small]
 & & & & \boxed{r+1} \arrow[dl, "r+1"'] & & & & \\
B_{\varpi_1}: \quad \boxed{\bar{1}} & \boxed{\bar{2}} \arrow[l, "1"'] & \cdots \arrow[l] & \boxed{\bar{r}} \arrow[l] & & \boxed{r} \arrow[ul, "r"'] \arrow[dl, "r+1"] & \cdots \arrow[l] & \boxed{2} \arrow[l] & \boxed{1} \arrow[l, "1"'] \\
 & & & & \boxed{\overline{r+1}} \arrow[ul, "r"] & & & &
\end{tikzcd}
$$

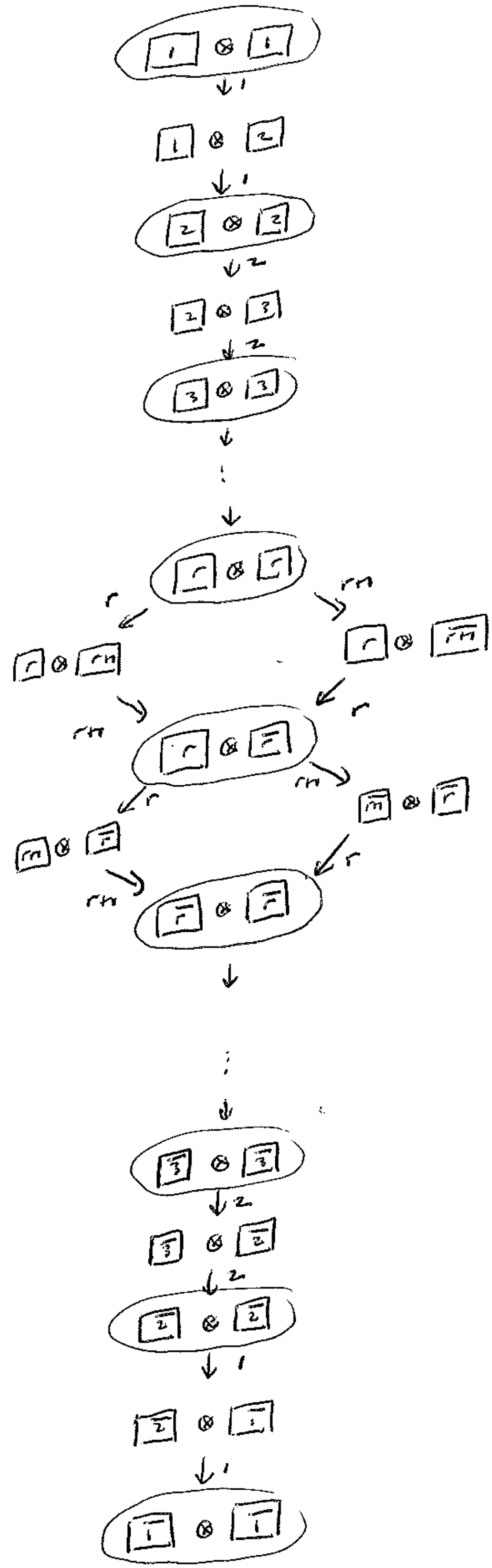
wt: $-e_1$, $-e_2$, $\ldots$, $-e_r$, e_{r+1} / $-e_{r+1}$, e_r, $\ldots$, e_2, e_1

Inside D_{r+1}-crystal

$$B_{\varpi_1} \otimes B_{\varpi_1}$$

we have

aligned elements circled



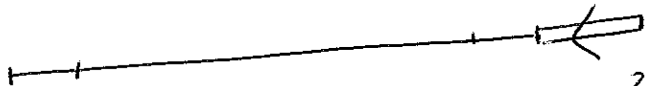
Above aligned elements support virtual D_{r+1}-crystal in $B_{\varpi_1} \otimes B_{\varpi_1}$:

$$
\begin{array}{ccc}
 & \widehat{wt} & wt \\
\boxed{1} \otimes \boxed{1} & 2e_1 & e_1 \\
\downarrow 1 & & \\
\boxed{2} \otimes \boxed{2} & 2e_2 & e_2 \\
\downarrow 2 & & \\
\boxed{3} \otimes \boxed{3} & 2e_3 & e_3 \\
\downarrow & \vdots & \\
\vdots & & \\
\downarrow & & \\
\boxed{r} \otimes \boxed{r} & 2e_r & e_r \\
\downarrow r & & \\
\boxed{r} \otimes \boxed{\bar{r}} & 0 & 0 \\
\downarrow r & & \\
\boxed{\bar{r}} \otimes \boxed{\bar{r}} & -2e_r & -e_r \\
\downarrow & \vdots & \vdots \\
\vdots & & \\
\downarrow & & \\
\boxed{\bar{3}} \otimes \boxed{\bar{3}} & -2e_3 & -e_3 \\
\downarrow 2 & & \\
\boxed{\bar{2}} \otimes \boxed{\bar{2}} & -2e_2 & -e_2 \\
\downarrow 1 & & \\
\boxed{\bar{1}} \otimes \boxed{\bar{1}} & -2e_1 & -e_1
\end{array}
$$

This is standard crystal B_{ϖ_1} for B_r

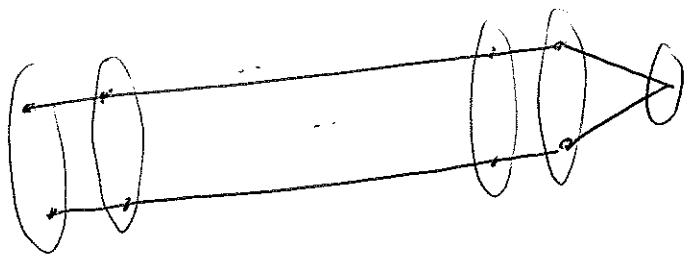
So far we discussed B_r vs D_{r+1}

We also have

C_r 

r_i: 1 1 . . . . 1 1 2

X

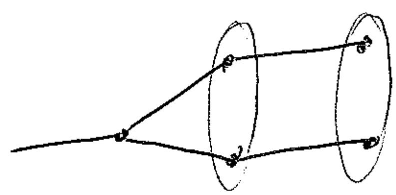
A_{2r-1} 

Y

F_4 

r_i: 2 2 1 1

X

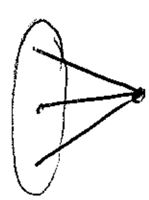
E_6 

Y

G_2 

r_i: 1 3

X

D_4 

Y

In all cases we use the semisimple and simply connected wt lattice for X, Y systems

Next we define virtual crystals for C_r using Stembridge crystals for A_{2r-1}

Recall

$\Phi = C_r$ $\quad\quad$ Λ $\quad\quad$ $Sp(2r)$

$\alpha_i = e_i - e_{i+1}$ $\quad 1 \le i \le r-1$

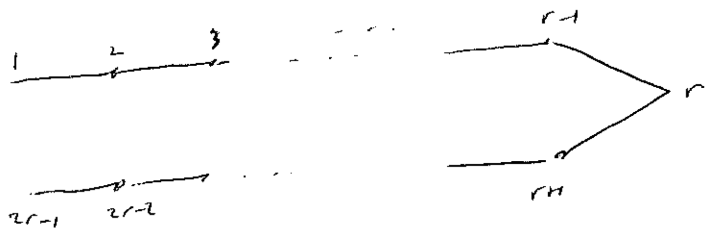
$\alpha_r = 2e_r$

$\bar{\omega}_i = e_1 + \dots + e_i$ $\quad 1 \le i \le r$

$$\Lambda = \sum_{i=1}^{r} \mathbb{Z}\bar{\omega}_i = \sum_{i=1}^{r} \mathbb{Z} e_i = \Lambda_{sc}$$

"the X system"

$\Phi = A_{2r-1}$ type $SL(2r)$



$\alpha_i = e_i - e_{i+1}$ $\quad 1 \le i \le 2r-1$

$\delta = e_1 + \cdots + e_{2r}$

$\varpi_i = e_1 + \cdots + e_i - \frac{i\delta}{2r}$

$$\Lambda = \sum_{i=1}^{2r-1} \mathbb{Z}\varpi_i = \left\{ \sum_{i=1}^{2r} a_i e_i \;\middle|\; a_i - a_j \in \mathbb{Z}, \ \sum_{i=1}^{2r} a_i = 0 \right\}$$
$$= \Lambda_{sc}$$

"the Y system"

We have

$$\psi: \begin{array}{lll} \Lambda^X & \longrightarrow & \Lambda^Y \\ \overline{\omega}_i^{\,X} & \longrightarrow & \overline{\omega}_i^{\,Y} + \overline{\omega}_{2r-i}^{\,Y} \qquad 1 \le i \le r-1 \\ \overline{\omega}_r^{\,X} & \longrightarrow & 2\overline{\omega}_r^{\,Y} \end{array}$$

ψ sends

$$e_i \longrightarrow e_i - e_{2r-i+1} \qquad 1 \le i \le r$$

$$\alpha_i^X \longrightarrow \alpha_i^Y + \alpha_{2r-i}^Y \qquad 1 \le i \le r-1$$

$$\alpha_r^X \longrightarrow 2\alpha_r^Y$$

Virtual operators are

$$e_i = \hat{e}_i \hat{e}_{2r-i} = \hat{e}_{2r-i} \hat{e}_i \qquad 1 \le i \le r-1$$

$$f_i = \hat{f}_i \hat{f}_{2r-i} = \hat{f}_{2r-i} \hat{f}_i$$

$$e_r = \hat{e}_r^{\,2}, \qquad f_r = \hat{f}_r^{\,2}$$

Given Stembridge crystal $\hat{B}$ for Y system.

An element $b \in \hat{B}$ is aligned whenever

$$\hat{\varphi}_i(b) = \hat{\varphi}_{2r-i}(b) \qquad 1 \le i \le r-1$$
$$\hat{\varepsilon}_i(b) = \hat{\varepsilon}_{2r-i}(b)$$

$\hat{\varphi}_r(b)$, $\hat{\varepsilon}_r(b)$ even

In this case define

$$\varphi_i(b) = \hat{\varphi}_i(b) = \hat{\varphi}_{2r-i}(b) \qquad 1 \le i \le r-1$$
$$\varepsilon_i(b) = \hat{\varepsilon}_i(b) = \hat{\varepsilon}_{2r-i}(b)$$

$$\varphi_r(b) = \frac{\hat{\varphi}_r(b)}{2}$$

$$\varepsilon_r(b) = \frac{\hat{\varepsilon}_r(b)}{2}$$

Ex $r=2$ C_2 vs A_3

Crystals for A_3

$$B_{\varpi_1}: \quad \boxed{4} \xleftarrow[3]{} \boxed{3} \xleftarrow[2]{} \boxed{2} \xleftarrow[1]{} \boxed{1}$$

wt $\quad \hat{e}_4 \quad \hat{e}_3 \quad \hat{e}_2 \quad \hat{e}_1$

$$\hat{e}_i = e_i - \frac{\iota\delta}{4}$$

$$B_{\varpi_3} \quad \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array} \xleftarrow[1]{} \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 4\\ \hline\end{array} \xleftarrow[2]{} \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \xleftarrow[3]{} \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$$

wt $\quad -\hat{e}_1 \quad -\hat{e}_2 \quad -\hat{e}_3 \quad \hat{e}_1+\hat{e}_2+\hat{e}_3 = -\hat{e}_4$

take

$$\hat{B} = B_{\varpi_1} \otimes B_{\varpi_3}$$

For $\hat{B}_1$ describe aligned elements and virtual operators.

Fn $\quad B\bar{\omega}_1$

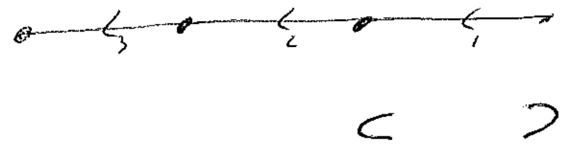
	$\boxed{4}$	$\boxed{3}$	$\boxed{2}$	$\boxed{1}$
φ_1	0	0	0	1
ε_1	0	0	1	0
φ_2	0	0	1	0
ε_2	0	1	0	0
φ_3	0	1	0	0
ε_3	1	0	0	0

Fn B_{ω_3}

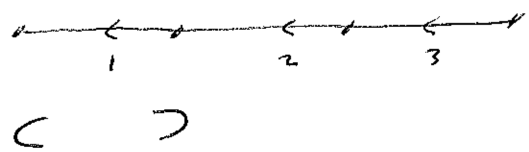
	$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$
φ_1	0	1	0	0
ε_1	1	0	0	0
φ_2	0	0	1	0
ε_2	0	1	0	0
φ_3	0	0	0	1
ε_3	0	0	1	0

Find $\hat{\varphi}_1$, $\hat{\varepsilon}_1$ for $\hat{B}$

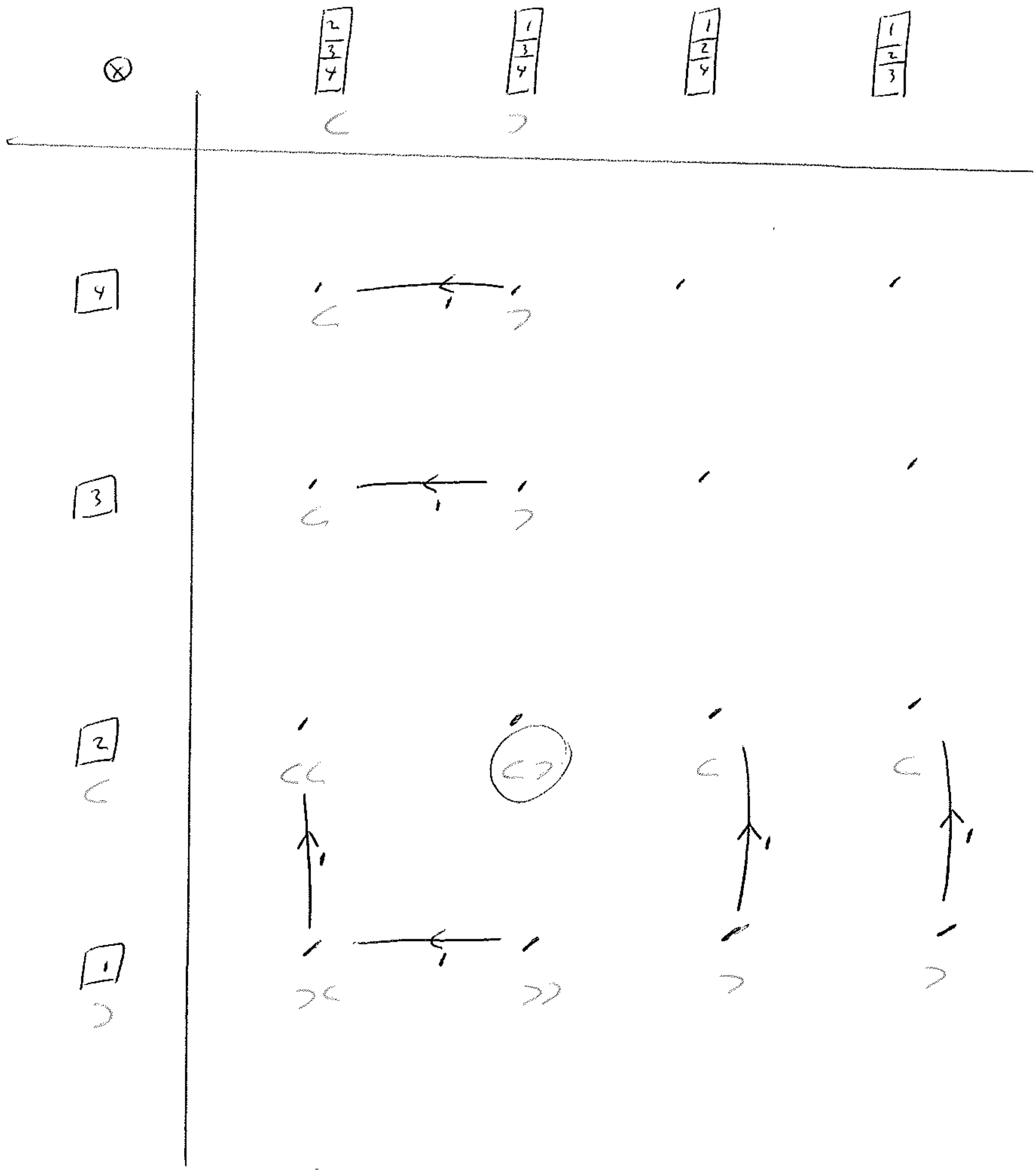
$B\bar{w}_1$



$B\bar{w}_3$



$\hat{B}$ A_3 $\hat{\varphi}_1$, $\hat{\varepsilon}_1$



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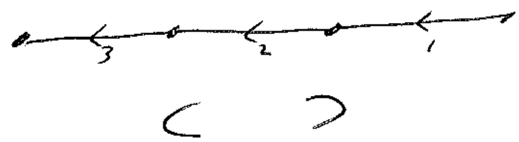
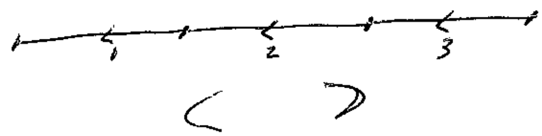
Find $\hat{\varphi}_2$, $\hat{\varepsilon}_2$ for $\hat{\beta}$

$\beta\bar{w}_1$



$\beta\bar{w}_3$



$\hat{B}$ A_3 $\hat{\varphi}_2$, $\hat{\varepsilon}_2$

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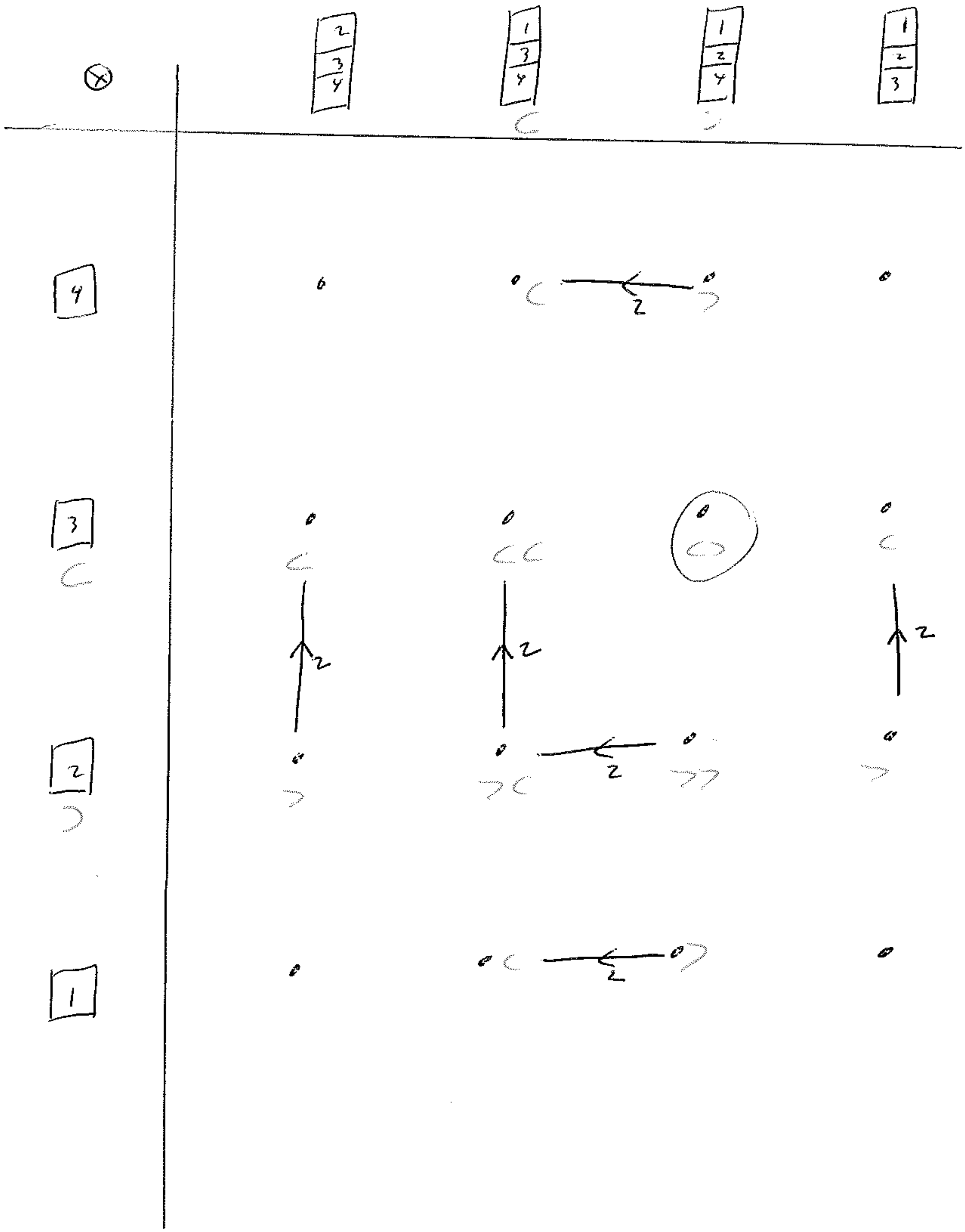
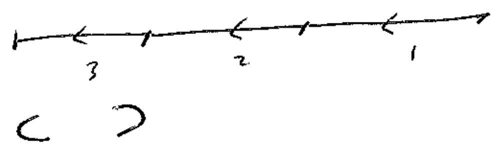
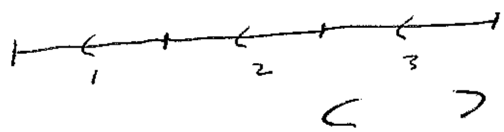


Find $\hat{\varphi}_3$, $\hat{\varepsilon}_3$ fn $\hat{B}$

$B\bar{w}_1$



$B\bar{w}_3$



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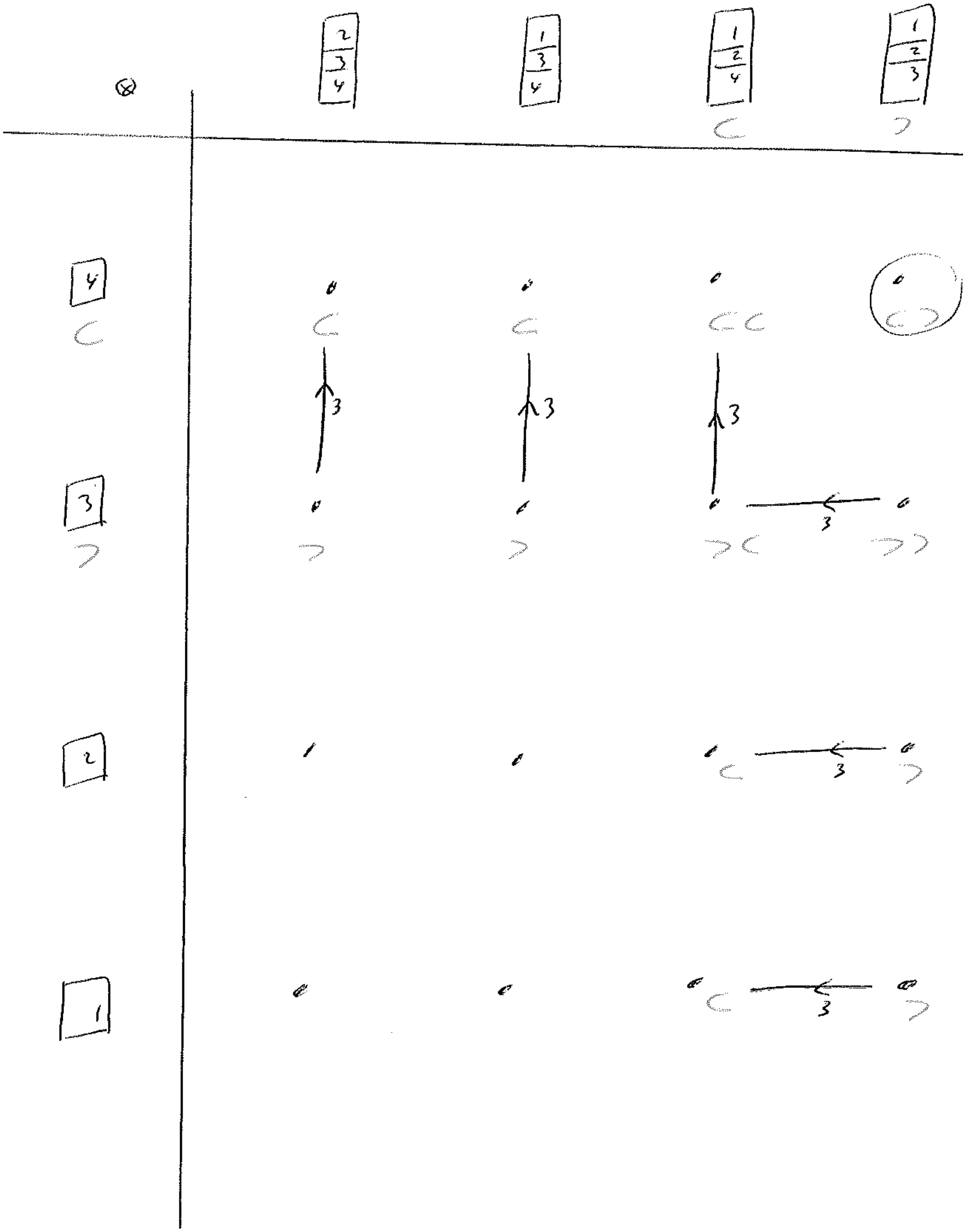
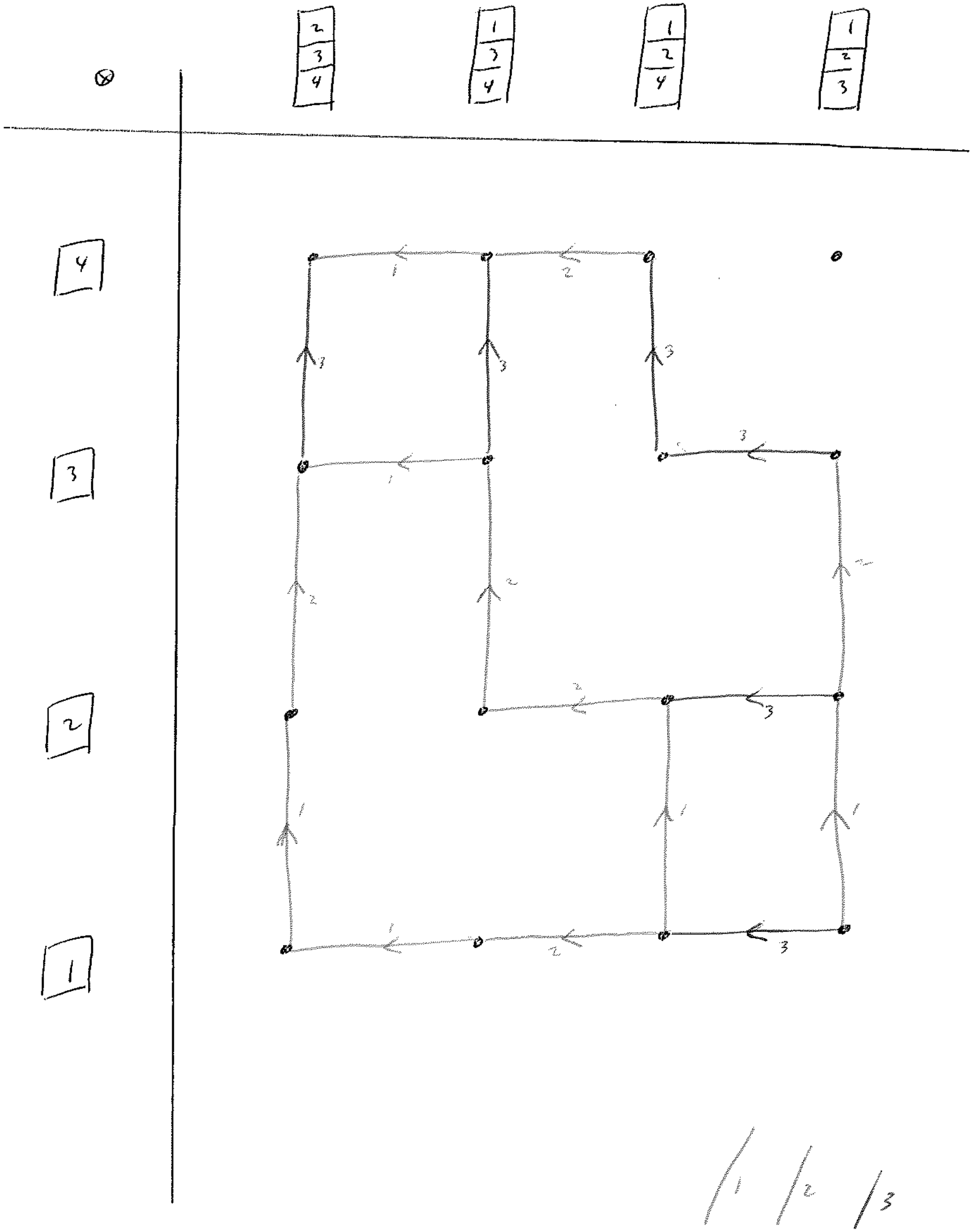
$\hat{\beta}$ A_3 $\hat{\varphi}_3$, $\hat{\varepsilon}_3$



$\hat{B}$ A_3 $\hat{\varphi}_i$, $\hat{\varepsilon}_i$ $i=1,2,3$



$\hat{B}$ A_3 circle aligned elements

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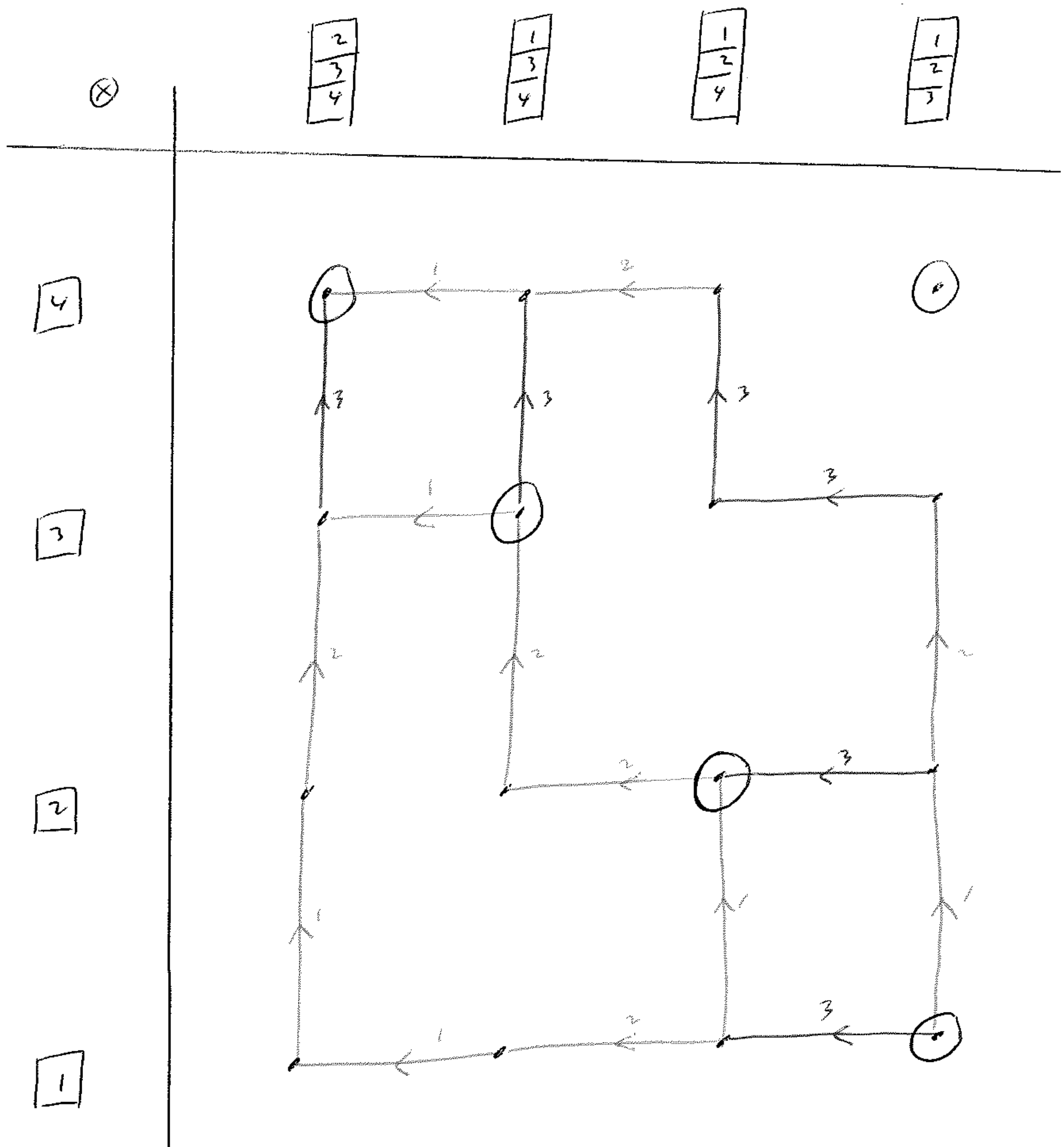

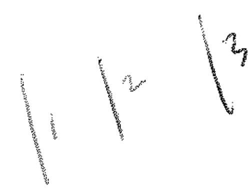

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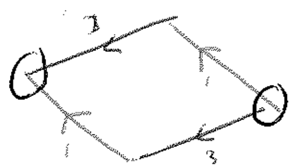
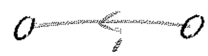
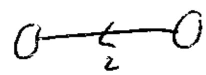
Describe virtual operators on the aligned elements

in $\hat{B}$

virtual op







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Describe virtual operators on the aligned elements

$$
\begin{array}{ccc}
 & \hat{wt} & wt \\
\boxed{1} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} & \hat{e}_1 - \hat{e}_4 = e_1 - e_4 & e_1 \\
\big\downarrow 1 & & \\
\boxed{2} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} & \hat{e}_2 - \hat{e}_3 = e_2 - e_3 & e_2 \\
\big\downarrow 2 & & \\
\boxed{3} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} & \hat{e}_3 - \hat{e}_2 = e_3 - e_2 & -e_2 \\
\big\downarrow 1 & & \\
\boxed{4} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} & \hat{e}_4 - \hat{e}_1 = e_4 - e_1 & -e_1
\end{array}
$$

This is the standard crystal for C_2

hw is $e_1 = \varpi_1$

call it B_{ϖ_1} for C_2

We have realized the crystals B_{ϖ_1} for C_2 as a virtual crystal for the A_3-crystal $B_{\varpi_1} \otimes B_{\varpi_3}$.

Similarly, for $r \geq 2$ the C_r-crystal B_{ϖ_1} is realized as a virtual crystal in the A_{2r-1}-crystal

$$B_{\varpi_1} \otimes B_{\varpi_{2r-1}}$$