Lecture 2 Friday Sept 6

For a root system Φ in V,

Fix a subspace $H \subseteq V$ with

$$\dim H = \dim V - 1$$

"hyperplane"

and

$$H \cap \Phi = \phi$$

Pick $\rho \in H^\perp$ and write

$$V = H + \mathbb{R}\rho$$

(orthog dir sum)

For $\alpha \in \Phi$ $\langle \alpha, \rho \rangle \neq 0$

Define

$$\Phi^+ = \{ \alpha \in \Phi \mid \langle \alpha, \rho \rangle > 0 \}$$ "pos roots"

$$\Phi^- = \{ \alpha \in \Phi \mid \langle \alpha, \rho \rangle < 0 \}$$ "neg roots"

A pos root α is <u>simple</u> whenever

$$\alpha \neq \alpha_1 + \alpha_2 \qquad \forall \alpha_1, \alpha_2 \in \Phi^+$$

Define

$$\Sigma = \{ \alpha \in \Phi^+ \mid \alpha \text{ simple} \}$$

LEM 3

(i) $\langle \alpha, \beta \rangle \leq 0$ for distinct $\alpha, \beta \in \Sigma$

(ii) The elements of Σ are linearly indep

(iii) Each $\alpha \in \Phi$ is a linear combin of Σ
with coefs in $\mathbb{N} = \{0, 1, 2, \ldots\}$

pf

(i) Suppose $\langle \alpha, \beta \rangle > 0$

$$\frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}, \quad \frac{2\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle}$$

are pos integers whose product is < 4

Swapping α, β WLOG

$$\frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} = 1$$

$$\Phi \ni \overset{\gamma}{\overset{\shortparallel}{r_\alpha(\beta)}} = \beta - \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

$$\gamma = \beta - \alpha$$

Case	Contradiction
$\gamma \in \Phi^+$	$\beta = \underset{+}{\alpha} + \underset{+}{\gamma}$
$\gamma \in \Phi^-$	$\alpha = \underset{+}{\beta} + \underset{+}{(-\gamma)}$

(ii) Suppose $\exists$ a linear dependency

$$0 = \sum_{\alpha \in \Sigma} a_\alpha \alpha \qquad a_\alpha \in \mathbb{R} \qquad a_\alpha \text{ not all } 0$$

Define

$$\Sigma^+ = \{ \alpha \in \Sigma \mid a_\alpha > 0 \}$$

$$\Sigma^- = \{ \alpha \in \Sigma \mid a_\alpha < 0 \}$$

WLOG $\Sigma^+ \neq \phi$.

Define

$$v = \sum_{\alpha \in \Sigma^+} a_\alpha \alpha$$

$$= \sum_{\beta \in \Sigma^-} -a_\beta \beta$$

obs

$$\langle v, \rho \rangle = \sum_{\alpha \in \Sigma^+} \overset{>0}{a_\alpha} \overset{>0}{\langle \alpha, \rho \rangle}$$

$$> 0$$

So $v \neq 0$

so

$$0 < \langle v, v \rangle = \left\langle \sum_{\alpha \in \Sigma^+} a_\alpha \alpha \, , \sum_{\beta \in \Sigma^-} -a_\beta \beta \right\rangle$$

$$= \sum_{\alpha \in \Sigma^+} \sum_{\beta \in \Sigma^-} \underset{\overset{\vee}{0}}{a_\alpha} \underset{\overset{\vee}{0}}{\underbrace{(-a_\beta)}} \underset{\overset{\wedge}{0}}{\langle \alpha, \beta \rangle}$$

$$\leq 0$$ Cnt.

(iii) Suppose $\exists$ counterexample α

of all CE, choose α such that

$\langle \rho, \alpha \rangle$ is minimal

By constr

$$\alpha \notin \Sigma$$

So $\alpha = \alpha_1 + \alpha_2$ $\qquad \alpha_1, \alpha_2 \in \Phi^+$

$$\langle \rho, \alpha \rangle = \underset{0}{\underset{\vee}{\langle \rho, \alpha_1 \rangle}} + \underset{0}{\underset{\vee}{\langle \rho, \alpha_2 \rangle}}$$

$$\langle \rho, \alpha_i \rangle < \langle \rho, \alpha \rangle \qquad i = 1, 2.$$

α_i not CE $\qquad i = 1, 2$

$\alpha = \alpha_1 + \alpha_2$ is not CE either, cont.

$\square$

For a root system Φ in V with simple roots Σ

Index Σ by set I

Write

$$\Sigma = \{\alpha_i\}_{i \in I}$$

For $i \in I$ write

$$s_i = r_{\alpha_i}$$

"simple reflection"

LEM 4 For $i \in I$, s_i sends

$$\alpha_i \to -\alpha_i$$

and permutes

$$\Phi^+ \setminus \{\alpha_i\}$$

Pf Given $\alpha \in \Phi \setminus \{\alpha_i\}$

Write $\alpha = \sum_{j \in I} a_j \alpha_j$ $\quad a_j \in \mathbb{N}$

$\alpha \neq \alpha_i$ so $\alpha \notin \mathbb{R}\alpha_i$ so $\exists\, j \in I \setminus i$ with $a_j > 0$

$$s_i(\alpha) = \alpha - \frac{2\langle \alpha_i, \alpha \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

So α, $s_i(\alpha)$ have same α_j-coef, which is positive

So $s_i(\alpha) \in \Phi^+$

Also $s_i(\alpha) \neq \alpha_i$ else $\alpha = -\alpha_i$

Result follows $\square$

For a root system Φ in V with simple roots Σ

Define
$$W = \text{subgroup of } O(V) \text{ gen by } \{r_\alpha\}_{\alpha \in \Phi}$$
"Weyl group"

Write
$$U = \text{Span}(\Phi)$$
so
$$V = U + U^\perp$$

For $\alpha \in \Phi$ $\quad r_\alpha = I$ on $U^\perp$

So
$$W \text{ acts as } I \text{ on } U^\perp$$

Elements of W permute Φ
$$|W| < \infty \text{ since } |\Phi| < \infty$$

Next show W is gen by $\{\sigma_i\}_{i \in I}$

LEM 5 Fn $0 \neq \alpha \in V$ and $\theta \in O(V)$

$$r_{\theta(\alpha)} = \theta r_\alpha \theta^{-1}$$

pf Fn $v \in V$

$$r_{\theta(\alpha)} v = v - \frac{2\langle \theta(\alpha), v \rangle}{\langle \theta(\alpha), \theta(\alpha) \rangle} \theta(\alpha)$$

$$= v - \frac{2\langle \alpha, \theta^{-1} v \rangle}{\langle \alpha, \alpha \rangle} \theta(\alpha)$$

Also

$$\theta r_\alpha \theta^{-1} v = \theta \left(\theta^{-1} v - \frac{2\langle \alpha, \theta^{-1} v \rangle}{\langle \alpha, \alpha \rangle} \alpha \right)$$

$$= v - \frac{2\langle \alpha, \theta^{-1} v \rangle}{\langle \alpha, \alpha \rangle} \theta(\alpha)$$

$\square$

LEM 6 the group W is gen by $\{s_i\}_{i \in I}$

pf let G = subgp of W gen by $\{s_i\}_{i \in I}$

show $G = W$

show $r_\alpha \in G \quad \forall \alpha \in \Phi^+$

Suppose $\exists$ counterexample α.

of all CE, choose α with $\langle \alpha, \rho \rangle$ minimal

By constr $\alpha \notin \Sigma$

For $i \in I$ $\quad$ *

$$s_i(\alpha) = \alpha - \frac{2\langle \alpha_i, \alpha \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

$$\in \Phi^+$$

Also

$$r_{s_i(\alpha)} = \underset{G}{\underset{\cap}{s_i}} \; \underset{G}{\underset{\not\cap}{r_\alpha}} \; \underset{G}{\underset{\cap}{s_i}} \quad \notin G$$

So $\quad s_i(\alpha)$ is CE

So $\quad \langle s_i(\alpha), \rho \rangle \geq \langle \alpha, \rho \rangle$

So by *

$$-2\frac{\langle \alpha_i, \alpha \rangle}{\langle \alpha_i, \alpha_i \rangle} \underbrace{\langle \alpha_i, \rho \rangle}_{> 0} \geq 0$$

So $\quad \langle \alpha_i, \alpha \rangle \leq 0$

Write

$$\alpha = \sum_{i \in I} a_i \alpha_i \qquad a_i \in \mathbb{N}$$

Obs

$$0 < \langle \alpha, \alpha \rangle = \sum_{i \in I} \overset{\overset{0}{\vee\!|}}{a_i} \overbrace{\langle \alpha_i, \alpha \rangle}^{\overset{0}{\wedge\!|}}$$

$$\leq 0$$

. cont.

Result follows. $\square$

For $i \in I$

$s_i^2 = 1$ $\quad$ s_i has order 2

For dist $i, j \in I$

define $n(i,j) =$ order of $s_i s_j$

So $(s_i s_j)^{n(i,j)} = 1$

<u>Aside</u> W has a presentation by gens $\{s_i\}_{i \in I}$ and relations

$s_i^2 = 1$ $\quad$ $i \in I$

$(s_i s_j)^{n(i,j)} = 1$ $\quad$ $i \neq j \in I$

" W is a Coxeter group "