Lecture 19 Wednesday October 16 10/16/19

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So, b is aligned whenever

$$\hat{\varphi}_i(b), \quad \hat{\varepsilon}_i(b) \quad \text{even} \qquad 1 \le i \le r-1$$

$$\hat{\varphi}_r(b) = \hat{\varphi}_{rn}(b)$$

$$\hat{\varepsilon}_r(b) = \hat{\varepsilon}_{rn}(b)$$

In this case

$$\varphi_i(b) = \frac{\hat{\varphi}_i(b)}{2} \qquad 1 \le i \le r-1$$

$$\varepsilon_i(b) = \frac{\hat{\varepsilon}_i(b)}{2}$$

$$\varphi_r(b) = \hat{\varphi}_r(b) = \hat{\varphi}_{rn}(b)$$

$$\varepsilon_r(b) = \hat{\varepsilon}_r(b) = \hat{\varepsilon}_{rn}(b)$$

For the moment, assume $\mathcal{B}$ consists of all the aligned $b \in \hat{\mathcal{B}}$.

check if $\mathcal{B}$ satisfies axioms A1, A2

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A1 : For $x, y \in B$ and $1 \le i \le r$

assume

$$\begin{tikzcd} x & y \arrow[l, "i"'] \end{tikzcd} \quad \text{in } B$$

show

$$wt(y) - wt(x) = \alpha_i^{X} \qquad *$$

$$\varphi_i(y) - \varphi_i(x) = 1 \qquad **$$

$$\varepsilon_i(x) - \varepsilon_i(y) = 1 \qquad ***$$

Case $1 \le i \le r-1$ We have

$$\begin{tikzcd} x & \bullet \arrow[l, "i"'] & y \arrow[l, "i"'] \end{tikzcd} \quad \text{in } \hat{B}$$

so

$$\begin{array}{ccccc} \hat{wt}(y) & - & \hat{wt}(x) & = & 2\alpha_i^{\vee} \\ \| & & \| & & \| \\ \Psi(wt(y)) & & \Psi(wt(x)) & & \Psi(\alpha_i^{x}) \end{array}$$

Ψ inj so $*$ holds

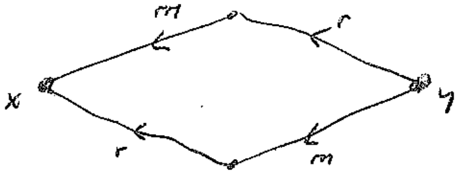
Also

$$\begin{array}{ccccc} \hat{\varphi}_i(y) & - & \hat{\varphi}_i(x) & = & 2 \\ \| & & \| & & \\ 2\varphi_i(y) & & 2\varphi_i(x) & & \end{array}$$

So $**$ holds.

Sim $***$ holds.

Case i=r We have



in $\hat{B}$

So

$$\begin{array}{ccccc} \hat{wt}(y) & - & \hat{wt}(x) & = & \underbrace{\alpha_r^\vee + \alpha_m^\vee} \\ \| & & \| & & \| \\ \psi(wt(y)) & & \psi(wt(x)) & & \psi(\alpha_r^{\times}) \end{array}$$

So *holds

Also

$$\begin{array}{cccc} \hat{\varphi}_r(y) & - & \hat{\varphi}_r(x) & = 1, \\ \| & & \| & \\ \varphi_r(y) & & \varphi_r(x) & \end{array} \qquad \begin{array}{cccc} \hat{\varphi}_m(y) & - & \hat{\varphi}_m(x) & = 1 \\ \| & & \| & \\ \varphi_r(y) & & \varphi_r(x) & \end{array}$$

So ** holds

Sim ***holds

A2 For $b \in B$ and $1 \leq i \leq r$

show

$$\langle wt(b), (\alpha_i^\vee)^X \rangle = \varphi_i(b) - \varepsilon_i(b)$$

This follows from

$$wt(b) = \sum_{i=1}^{r} (\varphi_i(b) - \varepsilon_i(b)) \bar{\omega}_i^X$$

Next we check that $B \cup \phi$ is closed under the virtual operators

For $1 \le i \le r$ consider α_i^X-root string

Case $1 \le i \le r-1$

String of even length:

in $\hat{B}$

$\hat{\varphi}_i$: 0 1 2 3 4 5 6

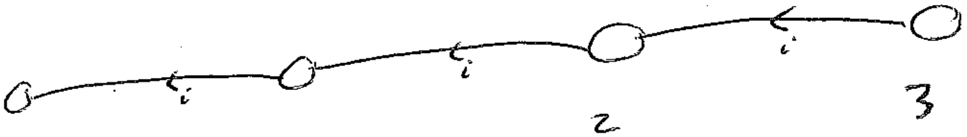
$\hat{\varepsilon}_i$: 6 5 4 3 2 1 0

For $b \in B$, require $\hat{\varphi}_i(b)$, $\hat{\varepsilon}_i(b)$ even.

Circle the aligned nodes:

in $\hat{B}$

Get



in B

φ_i: 0 1 2 3

ε_i: 3 2 1 0

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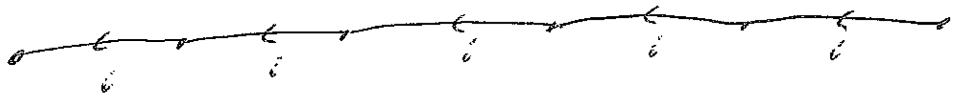
string of odd length:



in B

$\hat{\varphi}_i$	0	1	2	3	4	5
$\hat{\varepsilon}_i$	5	4	3	2	1	0

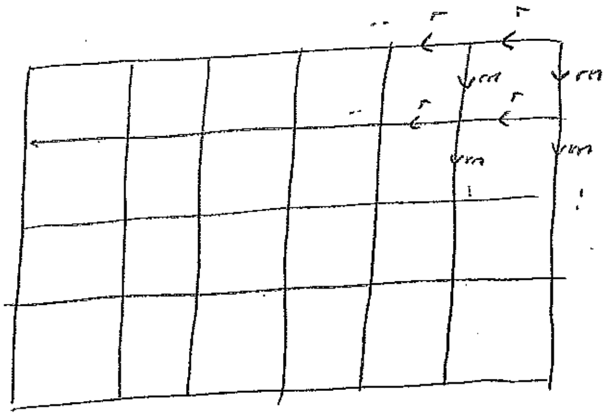
$\hat{\varphi}_i$, $\hat{\varepsilon}_i$ never both even.

none of these nodes aligned

Conclude $B \cup \emptyset$ closed under e_i, f_i

<u>Case $i=r$</u> Consider edges $\xleftarrow{r}$, $\xleftarrow{rn}$ in $\hat{B}$

Each connected component is a "rectangle"



in $\hat{B}$

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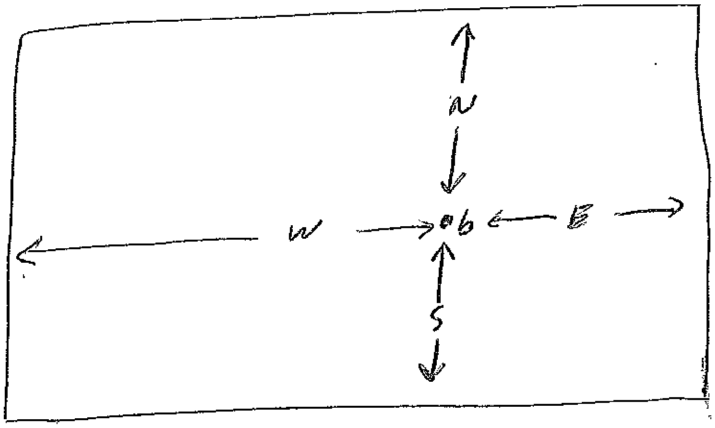
For a node b in above rectangle, describe

$\hat{\varphi}_r(b)$, $\hat{\varepsilon}_r(b)$, $\hat{\varphi}_{rn}(b)$, $\hat{\varepsilon}_{rn}(b)$



Since $\hat{B}$ is SemiNormal,

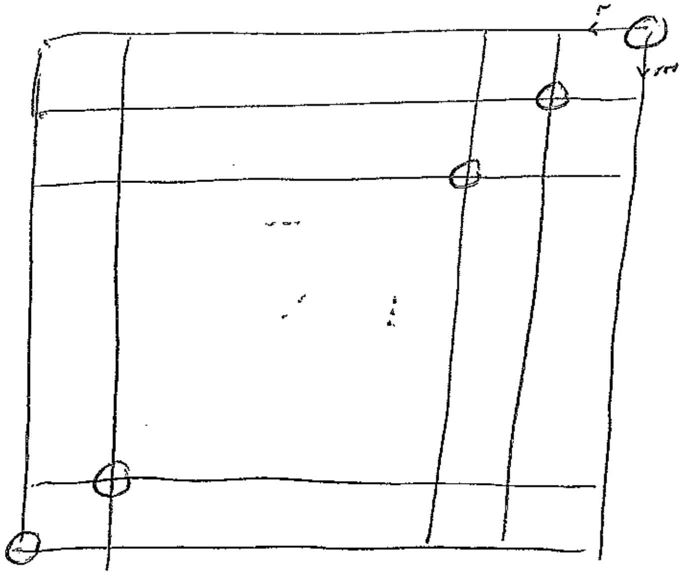
$\hat{\varphi}_r(b) = W$, $\hat{\varepsilon}_r(b) = E$

$\hat{\varphi}_{rn}(b) = S$, $\hat{\varepsilon}_{rn}(b) = N$

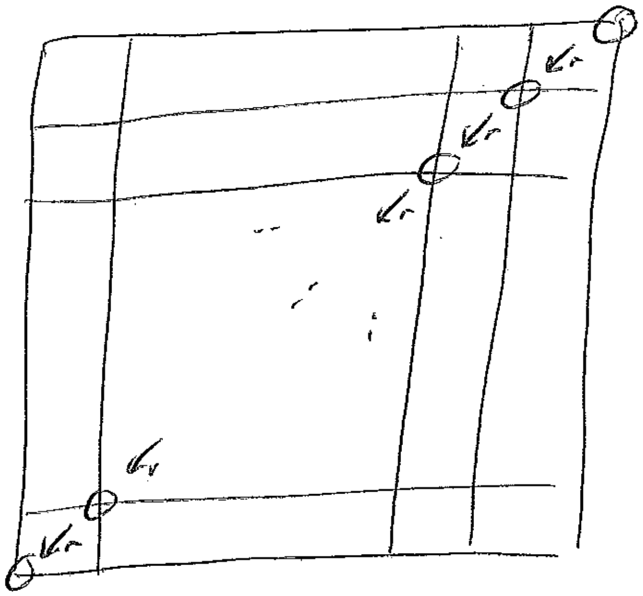
Fn $b \in B$ require

$\hat{\varphi}_r(b) = \hat{\varphi}_{rn}(b)$, $\hat{\varepsilon}_r(b) = \hat{\varepsilon}_{rn}(b)$

No nodes in rectangle are aligned unless it is a square, in which case the aligned nodes are circled below:



Showing $\xleftarrow[r]{}$ in B,



For each circled node b,

$$\varphi_r(b) = \hat{\varphi}_r(b) = \hat{\varphi}_{rm}(b)$$

$$\varepsilon_r(b) = \hat{\varepsilon}_r(b) = \hat{\varepsilon}_{rm}(b)$$

$B \cup \phi$ is closed under e_r, f_r

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We have shown that $B \cup \phi$ is closed under virtual ops, and is hence a crystal for the X system. By constr the crystal B is semi normal.

Note As we construct B, we do not require that B contain all the aligned elements. We only require that $B \cup \phi$ is closed under the virtual ops, and resulting crystal B is SN.

Def A virtual crystal (for the X system) is a nonempty subset $B \subseteq \hat{B}$ s.t

V1: $\hat{B}$ is Stembridge

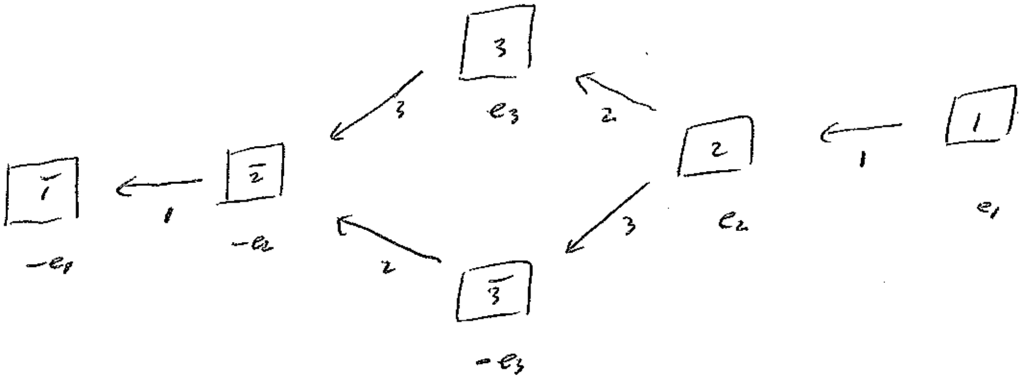
V2: each $b \in B$ is aligned

V3: $B \cup \phi$ is closed under virtual ops, and $\forall b \in B$

$$\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \phi\}, \quad \varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \phi\}$$

Ex $r=2$ B_2 vs D_3

Recall standard crystal for D_3:



wt

highest wt is

$$e_1 = \varpi_1$$

Call this crystal B_{ϖ_1}

Take

$$\hat{B} = B_{\varpi_1} \otimes B_{\varpi_1}$$

For $\hat{B}$,

describe the aligned elements and virtual ops.

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Fr $B\bar{w}_i$

$$
\begin{array}{c|cccccc}
 & \boxed{\bar{1}} & \boxed{\bar{2}} & \boxed{\bar{3}} & \boxed{3} & \boxed{2} & \boxed{1} \\
\hline
\varphi_1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\varepsilon_1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\varphi_2 & 0 & 0 & 1 & 0 & 1 & 0 \\
\varepsilon_2 & 0 & 1 & 0 & 1 & 0 & 0 \\
\varphi_3 & 0 & 0 & 0 & 1 & 1 & 0 \\
\varepsilon_3 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}
$$

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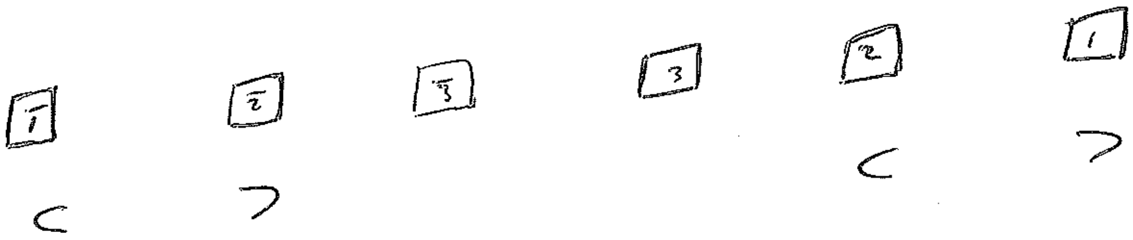
Find $\hat{\varphi}_1$, $\hat{\varepsilon}_1$ for $\hat{B}$



$$\underbrace{))\cdots)}_{\hat{\varphi}_i} \quad \underbrace{(\cdots(}_{\hat{\varepsilon}_i}$$

$\hat{B}$ D_3 $\hat{\varphi}_1, \hat{\varepsilon}_1$

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v3

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Find $\hat{\varphi}_2, \hat{\varepsilon}_2$ for $\hat{B}$

$$\boxed{\bar{1}} \quad \boxed{\bar{2}} \quad \boxed{\bar{3}} \quad \boxed{3} \quad \boxed{2} \quad \boxed{1}$$

$$(\quad) \quad (\quad)$$

$$\underbrace{)))}_{\hat{\varphi}_i} \quad \underbrace{(((}_{\hat{\varepsilon}_i}$$

$\hat{B}$ D_3 $\hat{\varphi}_2, \hat{\varepsilon}_2$

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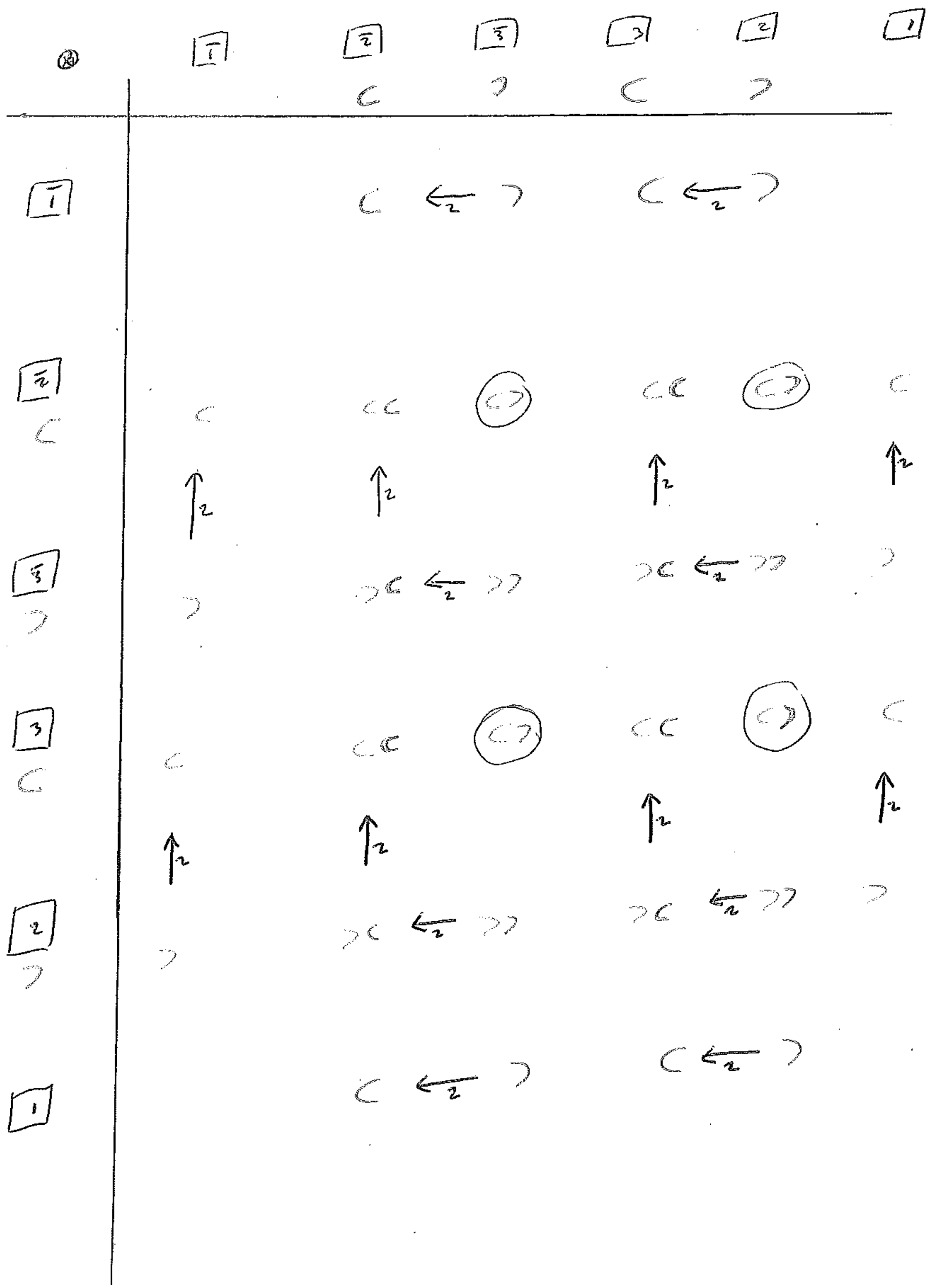

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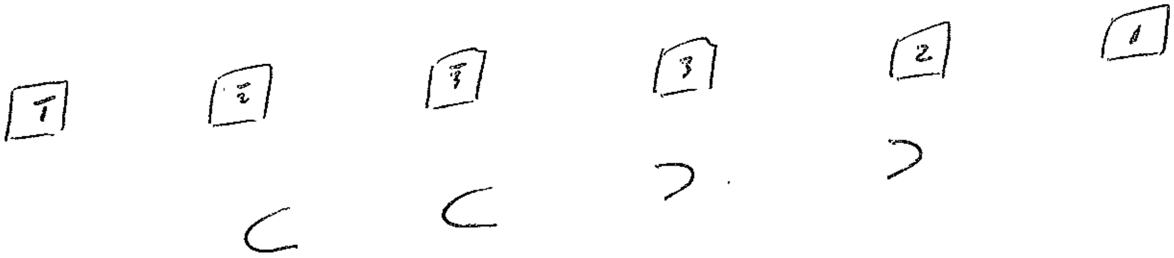
Find $\hat{\varphi}_3$, $\hat{\varepsilon}_3$ for $\hat{B}$



$$\underbrace{\supset\supset\supset}_{\hat{\varphi}_3} \quad \underbrace{\subset\subset\subset}_{\hat{\varepsilon}_3}$$

$\hat{B}$ D_3 $\hat{\varphi}_3, \hat{\varepsilon}_3$

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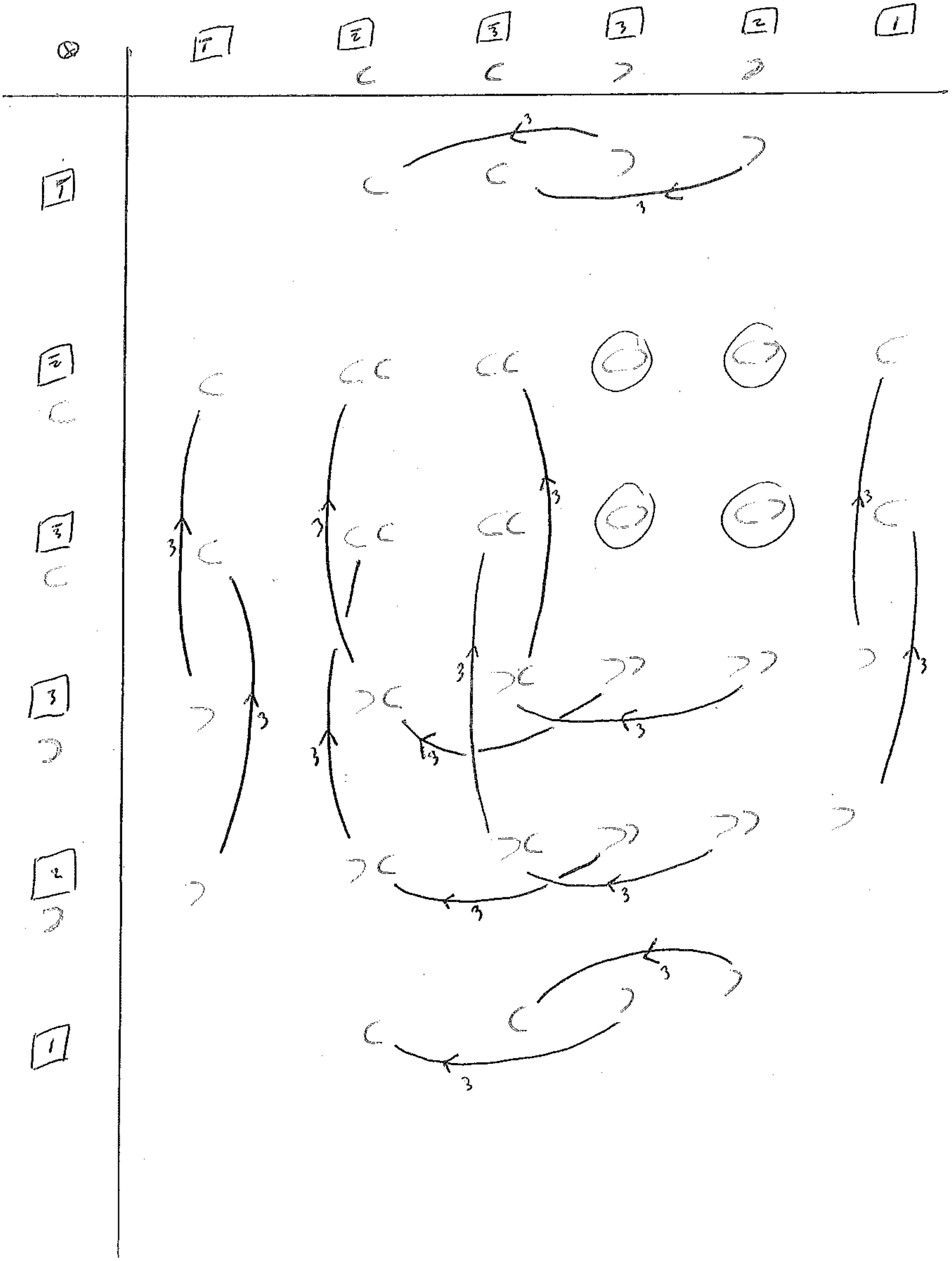

$\hat{B}$ D_3 $\hat{\varphi}_i$, $\hat{\varepsilon}_i$ $i=1,2,3$

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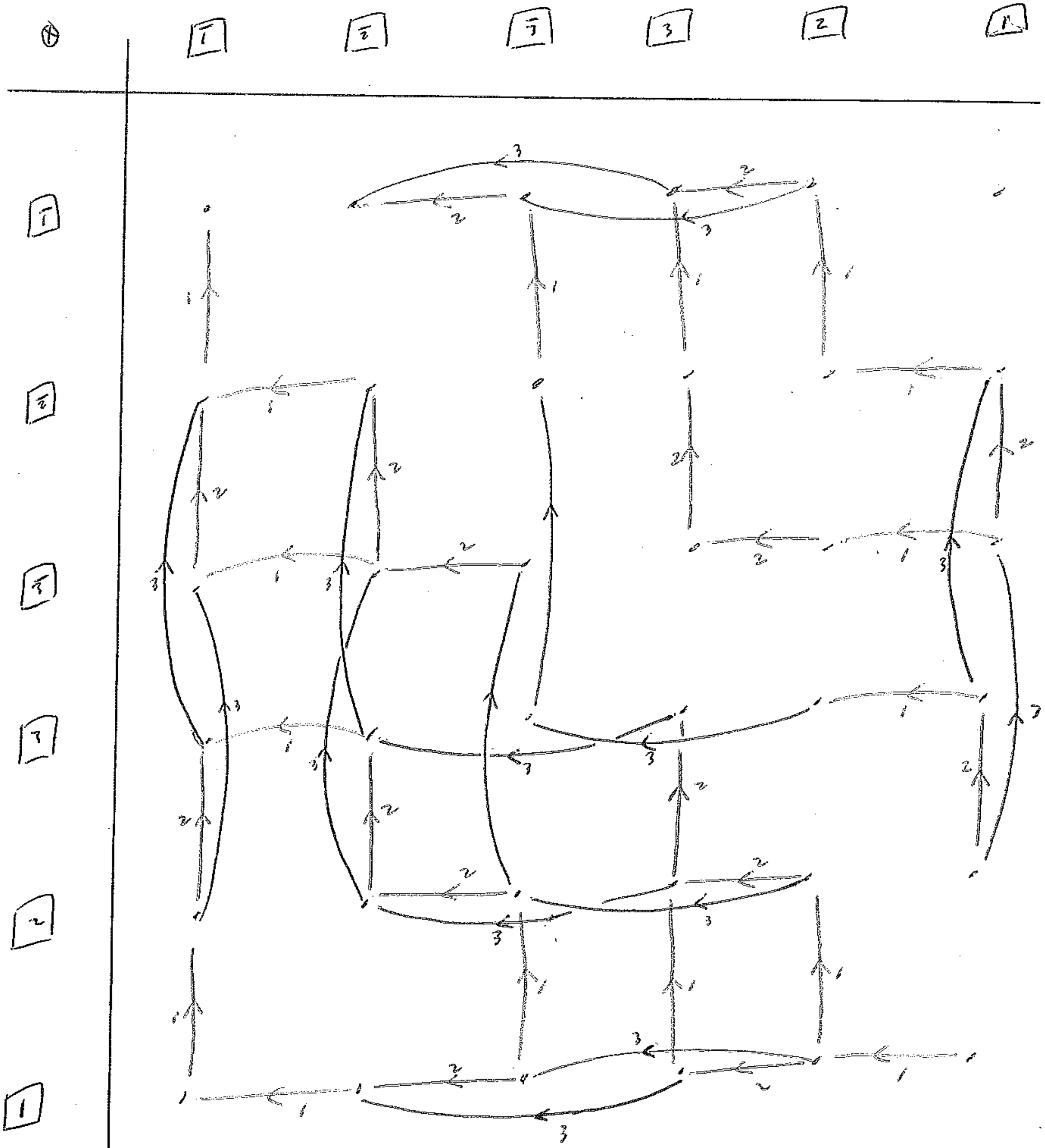

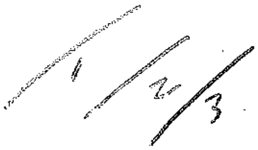

$\hat{B}$ D_3 circle aligned elements

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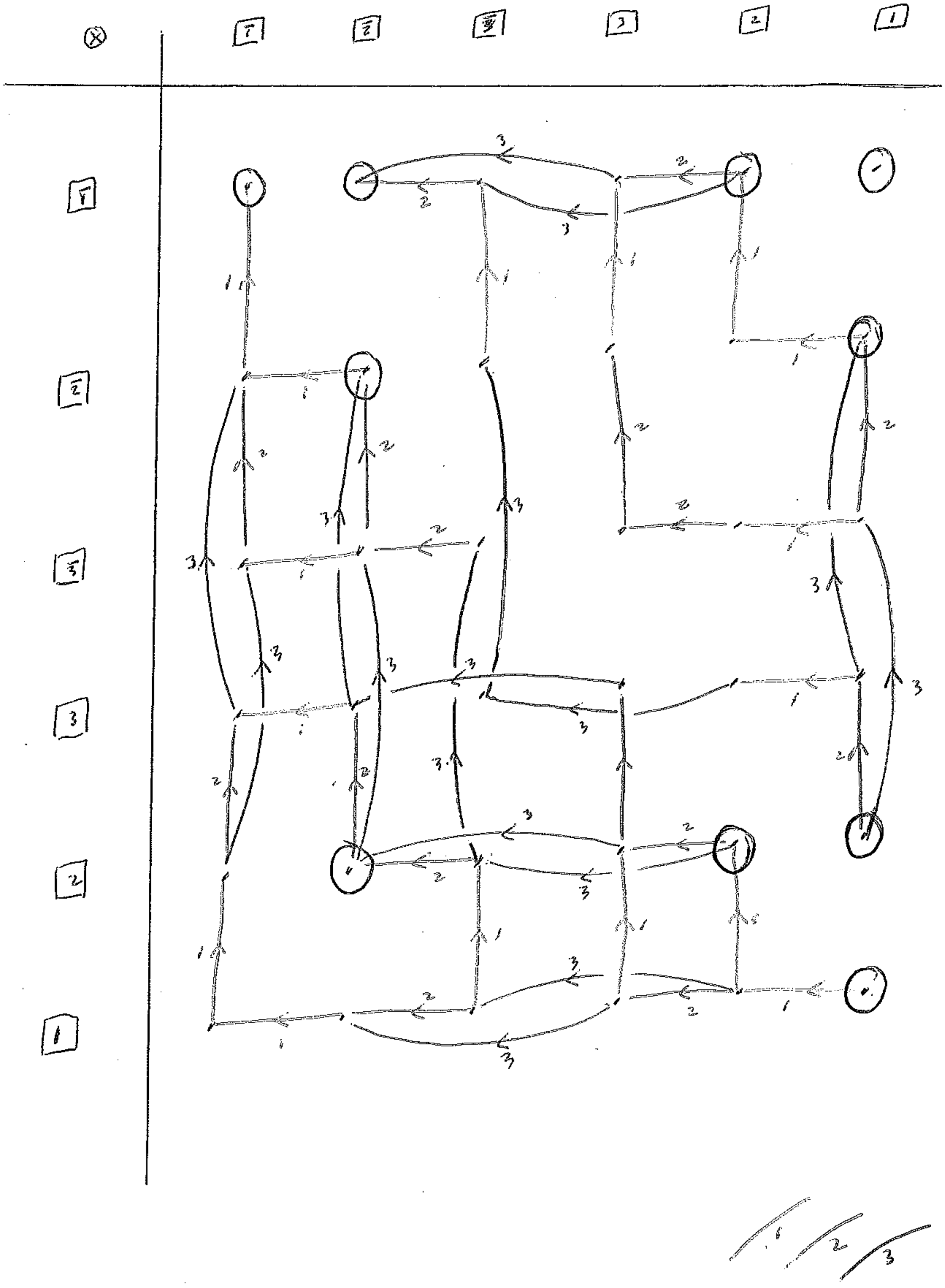

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Next describe the virtual operators on the aligned elements

in $\hat{\beta}$

virtual operator