Lecture 18 Monday October 14

LEM Given a connected Stembridge crystal B for a simply laced root system.

Given a hw element u for B.

Then $x \preceq u$ $\forall x \in B$.

pf Define

$$U = \{ x \in B \mid x \preceq u \}$$

show $U = B$

Assume $U \subsetneq B$

Since B is connected,

$\exists x \in U$ that is covered by an element in $B \setminus U$

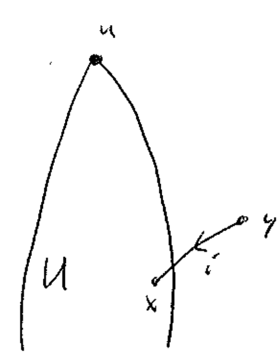
Of all such x, WLOG

$\langle \rho^\vee, wt(x) \rangle$ max'l $= M$ $\qquad \rho^\vee$ = dual Weyl vector

$x \neq u$ since u is hw

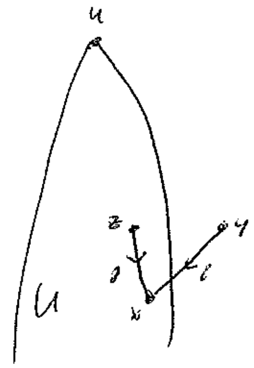
$\exists\, y \in B \setminus U$ $\exists\, i \in I$ st

$$x \xleftarrow{i} y$$



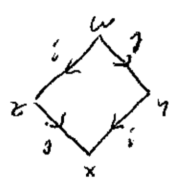
$\exists\ z \in \mathcal{U}\quad \exists j \in I$ st

$$x \xleftarrow{j} z$$

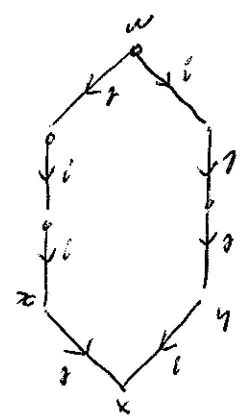


Obs $\langle \rho^\vee, wt(z) \rangle = M+1$

Since $\mathcal{B}$ is Stembridge, $\exists\ w \in \mathcal{B}$ st



or



$w \in \mathcal{U}$ by maximality of M

But $w \succeq y$

So $u \succeq w \succeq y$

Now $y \in \mathcal{U}$, cnt.

So $\mathcal{U} = \mathcal{B}$

$\square$

Thm Given a finite connected Stembridge crystal
B for a simply laced root system. then
B has a unique hw element u.

pf show u exists:
Recall dual Weyl vector $\rho^\vee$ satisfies
$$\langle \rho^\vee, \alpha_i \rangle = 1 \qquad \forall i \in I$$

Define $N = \max \left\{ \langle \rho^\vee, wt(x) \rangle \mid x \in B \right\}$

Pick $u \in B$ st
$$\langle \rho^\vee, wt(u) \rangle = N$$

$\forall v \in B,$
if v covers u in poset B
then $\langle \rho^\vee, v \rangle = N+1$
so v does not exist
so u is hw

We have shown u exists.
u is unique by prev lem. $\square$

Thm Given finite connected Stembridge crystals B, C for a simply laced root system.

TFAE

(i) B, C have same hw.

(ii) B, C are isomorphic.

pf Tedious see txt.

Thm For $I = A_r$ $GL(r+1)$ $n = r+1$

Given a finite connected Stembridge crystal B with hw λ. Then

(i) λ is a partition

(ii) crystals B, B_λ are isomorphic.

pf (i) write
$$\lambda = \sum_{i=1}^{n} \lambda_i e_i$$

Let u denote the hw element in B with $wt(u) = \lambda$

B is seminormal so
$$\varepsilon_i(u) = 0 \qquad i \in I$$

$$\begin{aligned}\langle \lambda, \alpha_i^\vee \rangle &= \langle wt(u), \alpha_i^\vee \rangle \\ &= \underbrace{\varphi_i(u)}_{\geq 0} - \underbrace{\varepsilon_i(u)}_{=0} \\ &\geq 0\end{aligned}$$

So $\lambda \in \Lambda^+$

So $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

λ is partition.

(ii) B_λ is finite, connected, Stembridge crystal with hw λ

So B, B_λ are iso by prev thm. $\square$

Consequences of Stembridge axioms

[Tedious proofs - I will illustrate with a crystal B_λ type A_2]

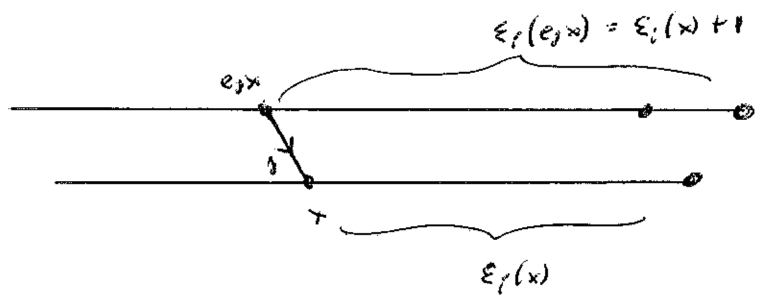
LEM Given a Stembridge crystal B for a simply-laced root system. Given $x \in B$ and distinct $i, j \in I$ st

$\langle \alpha_i, \alpha_j \rangle \neq 0$,
$\varepsilon_i(e_j x) = \varepsilon_i(x) + 1$.

Then

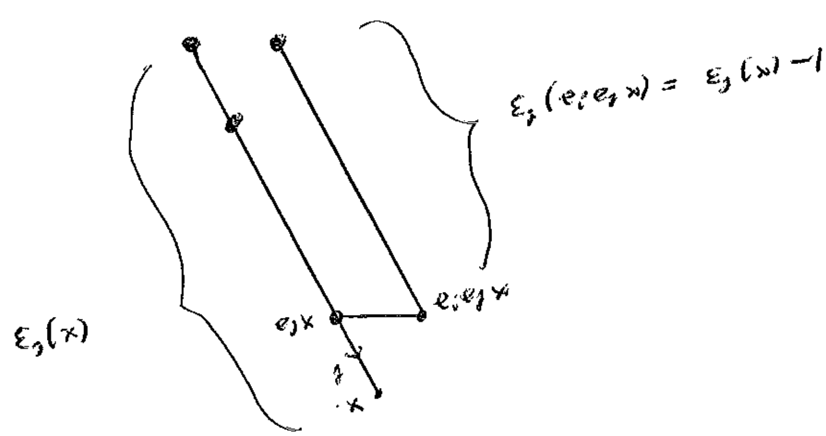
$$\varepsilon_j(e_i e_j x) = \varepsilon_j(x) - 1.$$

pf (for $\Phi = A_2$, $B = B_\lambda$) We assume



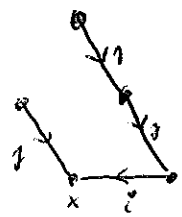
So level $x \leq 0$
So level $e_j x < 0$
$\therefore$ $e_i e_j x \leq 0$

So



LEM Given a Stembridge crystal B for a simply laced root system. Given $x \in B$ and distinct $i, j \in I$

st

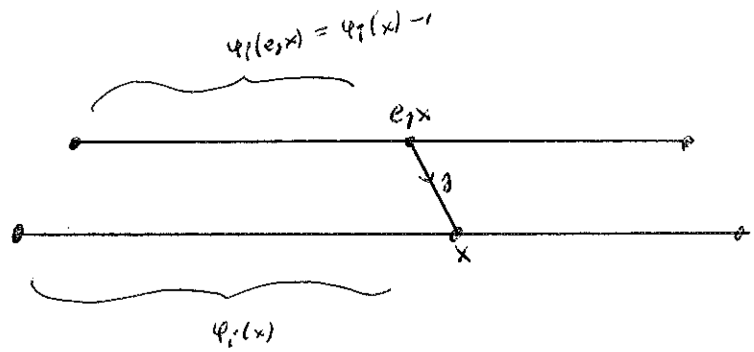


$\varphi_i(e_j x) < \varphi_i(x)$

Then

$$\varphi_i(e_j^2 e_i x) < \varphi_i(e_j e_i x)$$

pf (For $I = A_2$, $B = B_\lambda$)

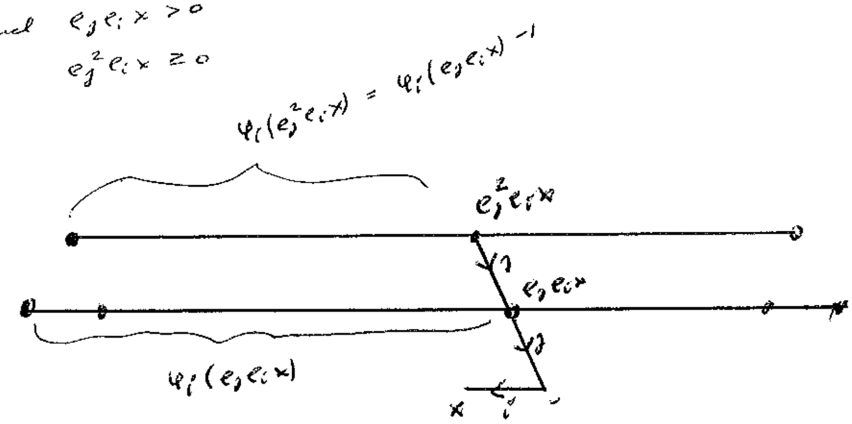


So level $x > 0$

So level $e_j e_i x > 0$

.. $e_j^2 e_i x \geq 0$

so



CH 5

In Ch 4 we defined Stembridge crystals for the simply-laced root systems.

We now do something similar for the remaining root systems

$$B_r, \quad C_r, \quad F_4, \quad G_2$$

First take

$$\Phi = B_r \qquad \Lambda \text{ spin}(2r+1) \text{ type}$$

To define our crystals for this data, we use Stembridge crystals for

$$D_{r+1}, \qquad \Lambda \text{ spin}(2r+2) \text{ type}$$

Recall

$\Phi = B_r$ $\qquad\qquad$ Λ $\quad$ spin$(2r+1)$

$\alpha_i = e_i - e_{i+1} \quad (1 \le i \le r-1),$ $\qquad\qquad$ $\alpha_i^\vee = \alpha_i$

$\alpha_r = e_r$ $\qquad\qquad$ $\alpha_r^\vee = 2e_r$

Fund wts

$\bar{\omega}_i = e_1 + \dots + e_i \qquad 1 \le i \le r-1$

$\bar{\omega}_r = \frac{e_1 + \dots + e_r}{2}$

$$\Lambda = \Lambda_{ac} = \sum_{i=1}^{r} \mathbb{Z}\bar{\omega}_i$$

$$= \left\{ \sum_{i=1}^{r} a_i e_i \;\middle|\; a_i \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{r} a_i e_i \quad a_i \in \mathbb{Z}_{odd} \right\}$$

"the X system"

$\Phi = D_{r+1}$ $\qquad\qquad$ Λ $\quad Spin(2r+2)$

$\alpha_i = e_i - e_{i+1}$ $\qquad 1 \le i \le r-1$

$\alpha_r = e_r - e_{r+1}$, $\qquad \alpha_{r+1} = e_r + e_{r+1}$

$\varpi_i = e_1 + \dots + e_i$ $\qquad 1 \le i \le r-1$

$$\varpi_r = \frac{e_1 + \dots + e_r - e_{r+1}}{2}$$

$$\varpi_{r+1} = \frac{e_1 + \dots + e_r + e_{r+1}}{2}$$

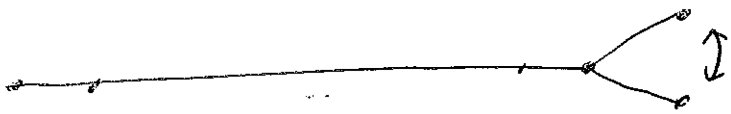
$$\Lambda = \Lambda_{sc} = \sum_{i=1}^{r+1} \mathbb{Z}\varpi_i$$

$$= \left\{ \sum_{i=1}^{r+1} a_i e_i \;\middle|\; a_i \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{r+1} a_i e_i \;\middle|\; a_i \in \mathbb{Z}_{odd} \right\}$$

"the Y system"

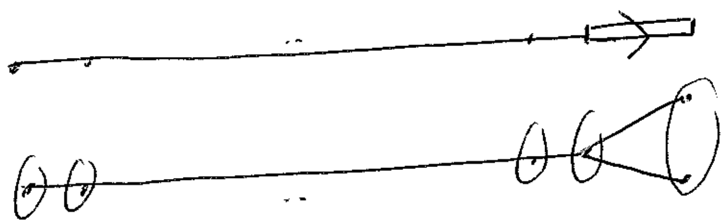
Note that both the X and Y system are semi simple and simply connected

Y system has a "diagram aut"



"aut"

orbits of aut are in bijection with I^X:



X

Y

For $i \in I^X$ define

$\sigma(i) =$ ith orbit for aut

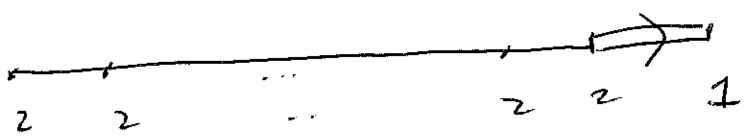
Also define

$$\gamma_i = \frac{\| \alpha_i^X \|^2}{\| \alpha \|^2}$$

where α is shortest root among

$\alpha_j^X \qquad j \in I^X$

So



γ_i

Define a group hom

$$\Psi : \Lambda^X \longrightarrow \Lambda^Y$$

$$\bar{\omega}_i^X \longrightarrow \gamma_i \sum_{j \in \sigma(i)} \bar{\omega}_j^Y \qquad i \in I^X$$

Thus Ψ sends

$$\bar{\omega}_i^X \longrightarrow 2\bar{\omega}_i^Y \qquad 1 \le i \le r-1$$

$$\bar{\omega}_r^X \longrightarrow \bar{\omega}_r^Y + \bar{\omega}_{r+1}^Y$$

One checks that Ψ sends

$$\alpha_i^X \longrightarrow \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y \qquad i \in I^X$$

So

$$\alpha_i^X \longrightarrow 2\alpha_i^Y \qquad 1 \le i \le r-1$$

$$\alpha_r^X \longrightarrow \alpha_r^Y + \alpha_{r+1}^Y$$

Map Ψ is inj.

Given a Stembridge crystal $\hat{B}$ for Y system

with functions

$$\hat{e}_i, \quad \hat{f}_i, \quad \hat{\varphi}_i, \quad \hat{\varepsilon}_i \quad \hat{wt}$$

For $i \in I^X$ define "virtual operators"

$$e_i = \prod_{j \in \sigma(i)} \hat{e}_j^{\,\gamma_i} \qquad f_i = \prod_{j \in \sigma(i)} \hat{f}_j^{\,\gamma_i}$$

So
$$e_i = \hat{e}_i^{\,2}, \qquad f_i = \hat{f}_i^{\,2} \qquad 1 \le i \le r-1$$

$$e_r = \hat{e}_r \hat{e}_{rn} = \hat{e}_{rn} \hat{e}_r$$
$$f_r = \hat{f}_r \hat{f}_{rn} = \hat{f}_{rn} \hat{f}_r$$

Given nonempty subset $B \subseteq \hat{B}$

Pick B st virtual operators turn B into a seminormal crystal for X-system.

Need
$$B \cup \{0\}$$
is closed under the virtual operators.

Also, need to define
$$wt, \quad \varphi_i, \quad \varepsilon_i$$

for B

The function

$$wt : \mathcal{B} \to \Lambda^X$$

should make this diag commute:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "wt"] \arrow[d, "\mathrm{incl}"'] & \Lambda^X \arrow[d, "\Psi"] \\
\hat{\mathcal{B}} \arrow[r, "\hat{wt}"'] & \Lambda^Y
\end{tikzcd}$$

wt lattice Λ^Y is semisimple,
so for $b \in \hat{\mathcal{B}}$,

$$\hat{wt}(b) = \sum_{i \in I^Y} \left(\hat{\varphi}_i(b) - \hat{\varepsilon}_i(b) \right) \varpi_i^Y$$

Also Λ^X is semisimple,
so for $b \in \mathcal{B}$ require

$$wt(b) = \sum_{i \in I^X} \left(\varphi_i(b) - \varepsilon_i(b) \right) \varpi_i^X \qquad \star$$

Apply Ψ to $\star$:

$$\psi(wt(b)) = \sum_{i \in I^X} \left(\varphi_i(b) - \varepsilon_i(b) \right) \psi(\overline{w_i}^X)$$

$$LHS = \hat{wt}(b)$$
$$= \sum_{j \in I^Y} \left(\hat{\varphi}_j(b) - \hat{\varepsilon}_j(b) \right) \overline{w_j}^Y$$

$$RHS = \sum_{i \in I^X} \left(\varphi_i(b) - \varepsilon_i(b) \right) \gamma_i \sum_{j \in \sigma(i)} \overline{w_j}^Y$$

Require that for $i \in I^X$,

$$\gamma_i \varphi_i(b) = \hat{\varphi}_j(b) \qquad \forall j \in \sigma(i)$$
$$\gamma_i \varepsilon_i(b) = \hat{\varepsilon}_j(b)$$

So we require

$$2\varphi_i(b) = \hat{\varphi}_i(b) \qquad 1 \leq i \leq r-1$$
$$2\varepsilon_i(b) = \hat{\varepsilon}_i(b)$$

$$\varphi_r(b) = \hat{\varphi}_r(b) = \hat{\varphi}_m(b)$$
$$\varepsilon_r(b) = \hat{\varepsilon}_r(b) = \hat{\varepsilon}_m(b)$$

What $b \in \hat{B}$ is allowed in B ?

Call b <u>aligned</u> whenever:

$\forall i \in I^{\times}$,

- $\hat{\varphi}_j(b)$ is indep of $j \in \sigma(i)$
  and this common value is divisible by γ_i
- $\hat{\varepsilon}_j(b)$ is indep of $j \in \sigma(i)$
  and this common value is divisible by γ_i

In this case define

$$\varphi_i(b) = \frac{\hat{\varphi}_j(b)}{\gamma_i} \qquad j \in \sigma(i)$$

$$\varepsilon_i(b) = \frac{\hat{\varepsilon}_j(b)}{\gamma_i}$$